MNRAS 000, 1–8 (2019) Preprint 24 January 2019 Compiled using MNRAS LATEX style file v3.0

A seismic scaling relation for stellar age

Earl Patrick Bellinger,[1]★
[1]SAC Postdoctoral Research Fellow, Stellar Astrophysics Centre, Aarhus, Denmark

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
A simple solar scaling relation for estimating the ages of solar-like stars from asteroseismic and spectroscopic data is developed. New seismic scaling relations for estimating mass and radius are presented as well, including a purely seismic radius scaling relation (i.e., no dependence on temperature). The relations show substantial improvement over the classical scaling relations and perform similarly well to grid-based modeling.

Key words: asteroseismology – stars: solar-type, fundamental parameters, evolution

1 INTRODUCTION

A solar scaling relation is a formula for estimating some unknown property of a star from observations by scaling from the known properties of the Sun. These relations have the form

$$\frac{Y}{\mathrm{Y}_\odot} \simeq \prod_i \left(\frac{X_i}{\mathrm{X}_{\odot,i}}\right)^{P_i} \tag{1}$$

where Y is some property we wish to estimate, such as the radius of the star. The quantity $\mathrm{Y}_\odot$ is the corresponding property of the Sun (e.g., the solar radius), the vector $\mathbf{X}$ contains measurable properties of the star (e.g., its effective temperature and luminosity), the vector $\mathbf{X}_\odot$ contains the corresponding solar properties, and $\mathbf{P}$ is some vector of exponents. An example scaling relation for estimating stellar radii can be derived from the Stefan–Boltzmann law as:

$$\frac{R}{\mathrm{R}_\odot} \simeq \left(\frac{T_{\mathrm{eff}}}{\mathrm{T}_{\mathrm{eff},\odot}}\right)^{-2} \left(\frac{L}{\mathrm{L}_\odot}\right)^{\frac{1}{2}} \tag{2}$$

where R is the radius, T_{eff} the effective temperature, and L the luminosity.

In the era of space asteroseismology, relations known as the *seismic scaling relations* have enjoyed wide usage. They are used to estimate the unknown masses and radii of stars by scaling asteroseismic observations with their helioseismic counterparts. These observations include the average frequency spacing between radial mode oscillations of consecutive radial order—the *large frequency separation*, $\Delta\nu$—and the *frequency at maximum oscillation power*, $\nu_{\max}$. From these and the observed effective temperature, one can estimate the stellar mass M and radius R of a star via:

$$\frac{M}{\mathrm{M}_\odot} \simeq \left(\frac{\nu_{\max}}{\nu_{\max,\odot}}\right)^{3} \left(\frac{\Delta\nu}{\Delta\nu_\odot}\right)^{-4} \left(\frac{T_{\mathrm{eff}}}{\mathrm{T}_{\mathrm{eff},\odot}}\right)^{\frac{3}{2}} \tag{3}$$

$$\frac{R}{\mathrm{R}_\odot} \simeq \left(\frac{\nu_{\max}}{\nu_{\max,\odot}}\right) \left(\frac{\Delta\nu}{\Delta\nu_\odot}\right)^{-2} \left(\frac{T_{\mathrm{eff}}}{\mathrm{T}_{\mathrm{eff},\odot}}\right)^{\frac{1}{2}} \tag{4}$$

where $\nu_{\max,\odot} = 3090 \pm 30$ μHz, $\Delta\nu_\odot = 135.1 \pm 0.1$ μHz, and $\mathrm{T}_{\mathrm{eff},\odot} = 5772.0 \pm 0.8$ K (Huber et al. 2011; Prša et al. 2016). These relations are especially useful thanks to the exquisite precision with which asteroseismic data can be obtained. For a typical well-observed solar-like star, $\Delta\nu$ and $\nu_{\max}$ can be measured with an estimated relative error of only 0.1% and 1%, respectively (see, e.g., Figure 5 of Bellinger et al. 2018).

The seismic scaling relations can be analytically derived from the fact that $\Delta\nu$ scales with the mean density of the star and $\nu_{\max}$ scales with the acoustic cut-off frequency (Ulrich 1986; Brown et al. 1991; Kjeldsen & Bedding 1995). They are not perfectly accurate, however, especially as one moves away from the main sequence (e.g., White et al. 2011; Gaulme et al. 2016; Guggenberger et al. 2016, 2017; Themeßl et al. 2018).

In recent years, there has been a great push for improved determination of stellar properties—and particularly stellar ages, for which asteroseismology is uniquely capable. Besides their intrinsic interest, knowledge of the ages of stars is useful for a broad spectrum of activities in astrophysics, ranging from charting the history of the Galaxy (e.g., Bland-Hawthorn & Gerhard 2016; Silva Aguirre et al. 2018) to understanding the processes of stellar and exoplanetary formation and evolution (e.g., Soderblom 2010; Winn & Fabrycky 2015; Nissen et al. 2017; Bellinger et al. 2017). Unfortunately, unlike with mass and radius, there is no scaling relation for stellar age. Instead, a multitude of methods have been developed to match asteroseismic observations of stars to theoretical models of stellar evolution (e.g., Brown et al. 1994; Pont & Eyer 2004; Lundkvist et al. 2014; Metcalfe et al. 2014; Lebreton & Goupil 2014; Silva Aguirre et al. 2015, 2017; Bellinger et al. 2016; Angelou et al. 2017; Rendle et al. 2019), which then yields their ages.

Scaling relations are attractive resources because they can easily and immediately be applied to observations with-

★ E-mail: bellinger@phys.au.dk

out requiring access to theoretical models. In this paper, I seek to develop such a relation to estimate the ages of stars, as well as to improve the scaling relations for estimating stellar mass and radius.

The strategy is as follows. It is by now well-known that the core-hydrogen abundance (and, by proxy, the age) of a main-sequence star is correlated with the average frequency spacing between radial and quadrupole oscillation modes (e.g., Christensen-Dalsgaard 1984; Aerts et al. 2010; Basu & Chaplin 2017). This spacing is known as the small frequency separation and is denoted by $\delta\nu$. The diagnostic power of $\delta\nu$ is owed to its sensitivity to the sound-speed gradient of the stellar core, which in turn is affected by the mean molecular weight, which increases monotonically over the main-sequence lifetime as a byproduct of hydrogen–helium fusion.

Here I formulate new scaling relations that make use of this spacing $\delta\nu$, and I also add a term for metallicity. Rather than by analytic derivation, I calibrate the exponents of this relation using 80 solar-type stars whose ages and other parameters have been previously determined through detailed fits to stellar evolution simulations. Finally, I perform cross-validation to estimate the accuracy of the new relations.

2 DATA

I obtained spectroscopic and asteroseismic measurements of 97 solar-like stars from the *Kepler* Ages (Silva Aguirre et al. 2015; Davies et al. 2016) and LEGACY (Lund et al. 2017; Silva Aguirre et al. 2017) samples, which were observed by the *Kepler* spacecraft during its nominal mission (Borucki et al. 2010). The ages, masses, and radii of these stars are taken from Bellinger et al. (2018) as derived using the *Stellar Parameters in an Instant* (SPI, Bellinger et al. 2016) pipeline. The SPI method uses machine learning to rapidly compute stellar ages, masses, and radii of stars by connecting their observations to theoretical models. For the present study, I selected 80 of these stars having $\delta\nu$ measurements with uncertainties smaller than 10% and $\nu_{\max}$ measurements with uncertainties smaller than 5%.

3 METHODS

Here I will detail the construction of the new scaling relations and the procedure for their calibration. The scaling relations shown in Equations 3 and 4 can be written more generically as follows. For a given quantity Y (and corresponding solar quantity $\mathrm{Y}_\odot$),

$$\frac{Y}{\mathrm{Y}_\odot} \simeq \left(\frac{\nu_{\max}}{\nu_{\max,\odot}}\right)^{\alpha} \left(\frac{\Delta\nu}{\Delta\nu_\odot}\right)^{\beta} \left(\frac{\delta\nu}{\delta\nu_\odot}\right)^{\gamma} \left(\frac{T_{\mathrm{eff}}}{T_{\mathrm{eff},\odot}}\right)^{\delta} \exp\left([\mathrm{Fe/H}]\right)^{\epsilon} \quad (5)$$

for suitable choices of the powers $\mathbf{P} = [\alpha, \beta, \gamma, \delta, \epsilon]$. Note the metallicity is first exponentiated. The uncertainties on all solar quantities are propagated except in the case of metallicity, where there is no agreed upon uncertainty (e.g., Bergemann & Serenelli 2014). From an analysis of solar data (Davies et al. 2014) one can find $\delta\nu_\odot = 8.957 \pm 0.059$ μHz. For the solar age I use $\tau_\odot = 4.569 \pm 0.006$ Gyr (Bonanno & Fröhlich 2015). To give a concrete example, in order to recover the classical radius scaling relation (Equation 4), i.e., $Y = R$, we have $\mathbf{P} = [1, -2, 0, 1/2, 0]$.

Table 1. Classical and MCMC-fitted exponents for scaling relations (see Equation 5).

		$\nu_{\max}$	$\Delta\nu$	$\delta\nu$	T_{eff}	[Fe/H]
	Y	α	β	γ	δ	ϵ
Classic	M	3	-4	–	1.5	–
New	M	0.975	-1.435	–	1.216	0.270
Classic	R	1	-2	–	0.5	–
New	R	0.305	-1.129	–	0.312	0.100
Seismic	R	0.883	-1.859	–	–	–
New	Age	-6.556	9.059	-1.292	-4.245	-0.426

I now seek to find the exponents for scaling relations that best match to the literature values of radius, mass, and age. I define the goodness-of-fit χ^2 for a given vector $\mathbf{P}$ as

$$\chi^2 = \sum_i \left(\frac{\hat{Y}_i - Y_i}{\sigma_i}\right)^2 \quad (6)$$

where $\hat{Y}_i$ is the literature value of the desired quantity (e.g., age) for the ith star, Y_i is the result of applying the scaling relation in Equation 5 with the given powers $\mathbf{P}$, and the uncertainties $\boldsymbol{\sigma}$ are the measurement uncertainty of $\hat{\mathbf{Y}}$. Given this function, I use Markov chain Monte Carlo (Foreman-Mackey et al. 2013; Foreman-Mackey 2016) to determine the posterior distribution of $\mathbf{P}$ for each of $Y = R, M, \tau$. Finally, I use the mean-shift algorithm (Fukunaga & Hostetler 1975; Pedregosa et al. 2011) to find the mode of the posterior distribution (see Figure 1). This yields the desired exponents for each scaling law.

4 RESULTS

The best exponents for each of the computed scaling relations are shown in Table 1. I fit the mass and radius scaling relations both with and without a dependence on $\delta\nu$. I found that its inclusion made little difference, however, so I have omitted it for those two relations.

A striking feature of the new radius scaling relation is its small dependence on spectroscopic variables. To explore this further, I have additionally fit a radius scaling relation using only $\Delta\nu$ and $\nu_{\max}$. We shall soon see in the next section that although it is not quite as accurate as the full new radius relation, it does outperform the classical radius scaling relation, despite requiring no spectroscopic information.

A notable aspect of the new relations is that their exponents are smaller in magnitude than those of the classical relations. As a consequence, the resulting uncertainties are also smaller. We may estimate the typical uncertainties of applying these relations by examining the typical uncertainties of their inputs. The median relative errors on $\nu_{\max}$, $\Delta\nu$, $\delta\nu$, T_{eff}, and $\exp$ [Fe/H] of the *Kepler* Ages and LEGACY stars are approximately 1%, 0.1%, 4%, 1%, and 10%, respectively (e.g., Figure 5 of Bellinger et al. 2018). Thus, for a solar twin observed with such uncertainties, using Equation 5 with the exponents given in Table 1 yields uncertainties of 0.032 $\mathrm{M}_\odot$ (3.3%), 0.011 $\mathrm{R}_\odot$ (1.1%), and 0.56 Gyr (12%). These values are similar to those from fits to theoretical models (e.g., Figure 6 of Bellinger et al. 2018).

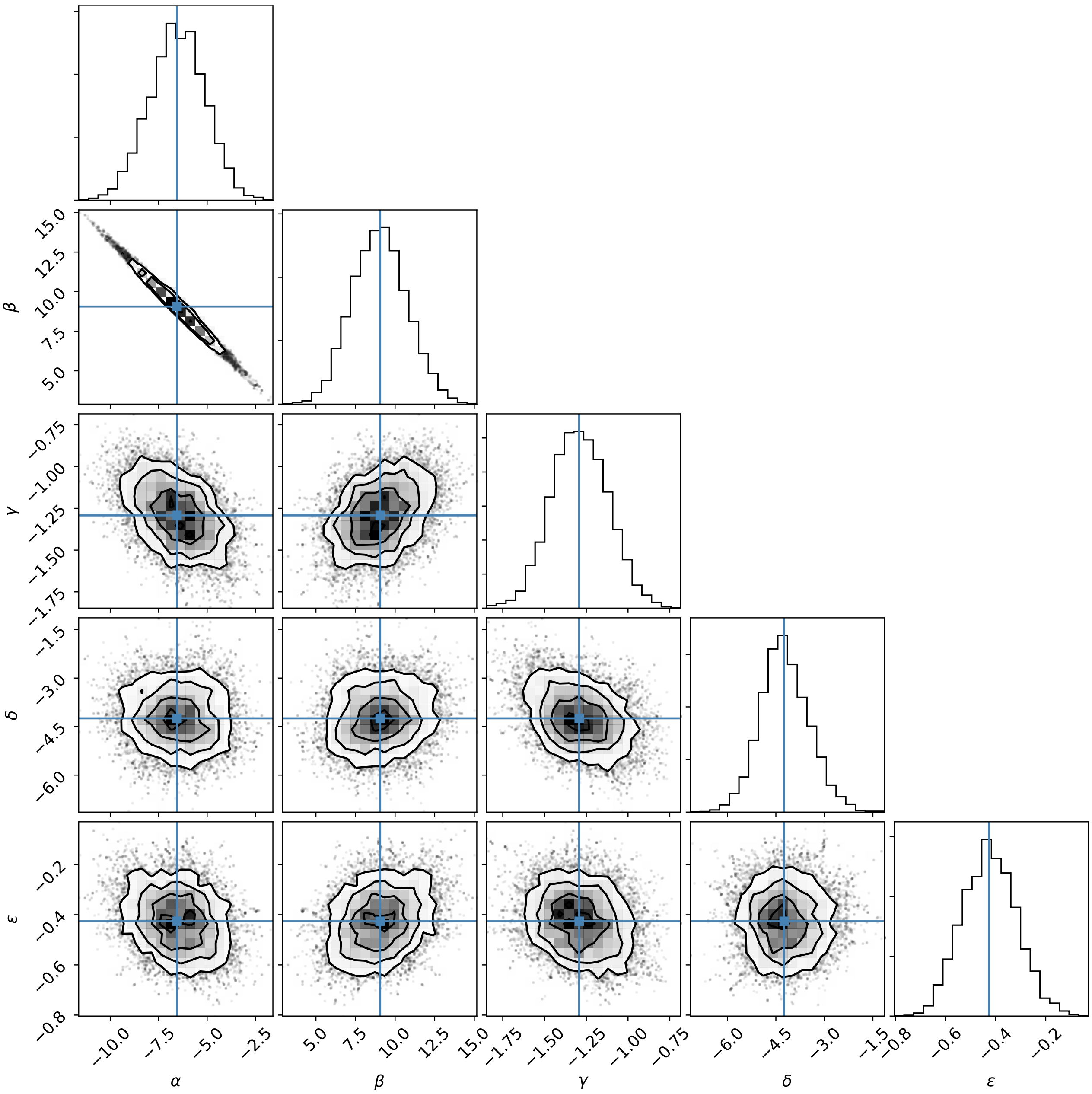

Figure 1. A standard corner plot showing the posterior distributions of each of the exponents in the MCMC-fitting of the age scaling relation (*cf.* Equation 5). Since $\nu_{\max}$ and $\Delta\nu$ are correlated quantities (e.g., Stello et al. 2009), the respective exponents for these quantities (α and β) are correlated as well. The blue lines and points indicate the mode of the posterior distribution, which are listed in Table 1.

5 BENCHMARKING

I now seek to determine the accuracy of the relations: i.e., how well do these relations actually work? The fitted values cannot simply be compared to the literature values: it would be unsurprising if they match, being that they were numerically optimized to do so. Instead, we may use cross-validation to answer this question. The procedure is as follows. We take our same data set as before, but instead of training on the entire data set, we remove one of the stars. We then fit the relations using the other 79 stars that were not held out using the procedure described in the previous section. Finally, the newly fitted relations are tested on the held-out star. This test is then repeated for every star.

Comparisons of these cross-validated relations to literature values are shown in Figures 2, 3, and 4. The classical mass and radius scaling relations are also shown, and there it can be seen that the new relations are better at reproducing the literature values. The age scaling relation is also compared to the BASTA ages (Silva Aguirre et al. 2015, 2017), which were fit in a different way and using a different grid of theoretical models. The age scaling relation shows a larger dispersion at older ages, which is most likely due to the input data having larger uncertainties there (see Figure 5). The purely seismic radius scaling relation is shown in Figure 6. This relation also outperforms the classical scaling relation, despite having no temperature dependence.

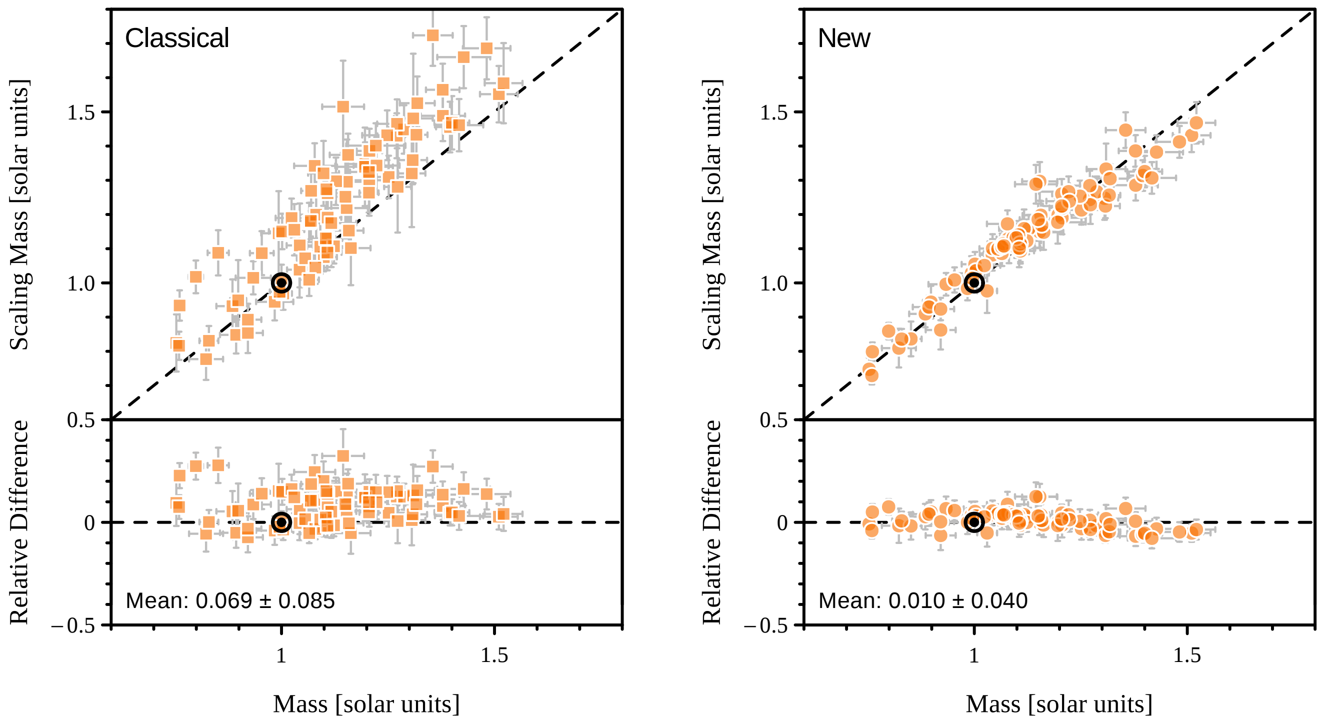

Figure 2. Classical (left, squares) and new (right, circles) scaling relations for estimating stellar mass. Each point is a star observed by *Kepler*. The masses on the x-axis the literature values; they were estimated using the SPI method with reference to a grid of theoretical stellar models. The masses on the y-axis have been estimated using the scaling relation (whose fitting did not involve the star being plotted, see Section 5 for details). The bottom panel shows the relative difference between the scaling mass and the literature mass. The weighted mean and standard deviation of the ratios are given. The Sun is denoted by the solar symbol (⊙).

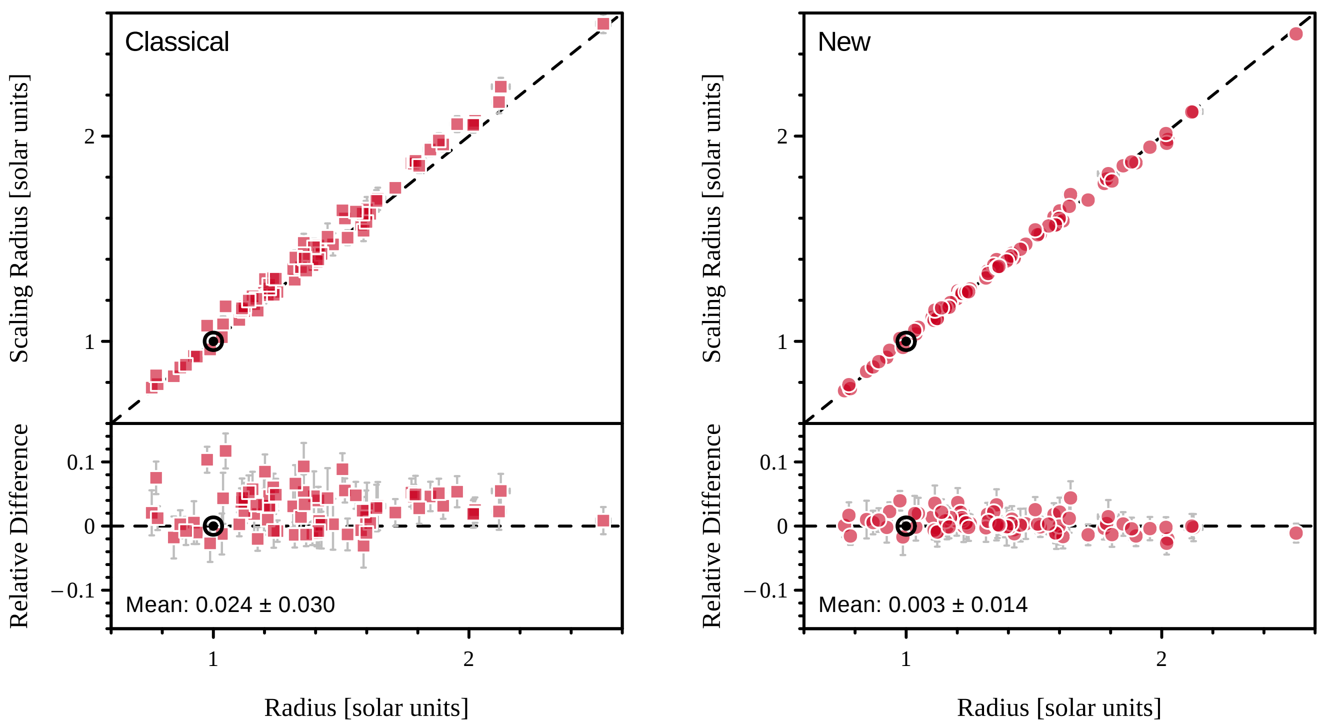

Figure 3. Classical (left, squares) and new (right, circles) scaling relations for estimating stellar radius.

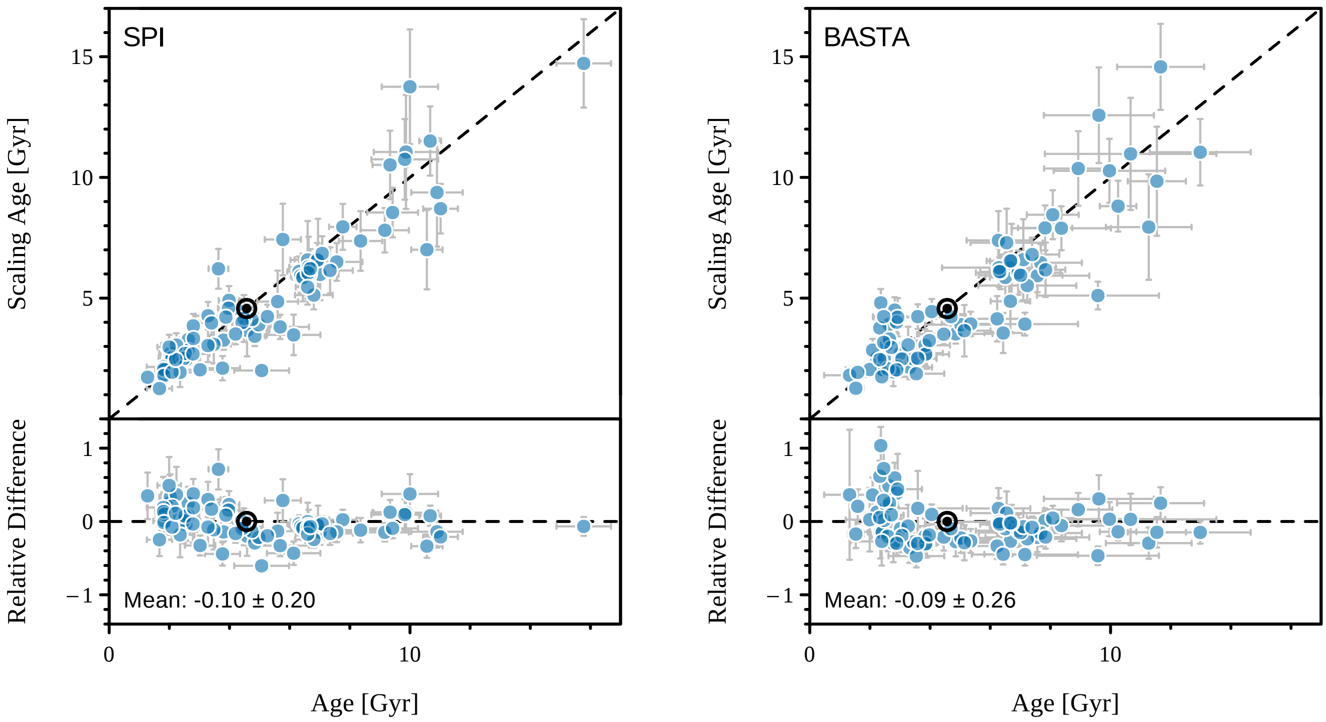

Figure 4. A comparison of the scaling relation for stellar age to literature age values. Left panel: cross-validated scaling ages compared to literature values from SPI. Right panel: scaling ages using the values from Table 1 in comparison with BASTA ages.

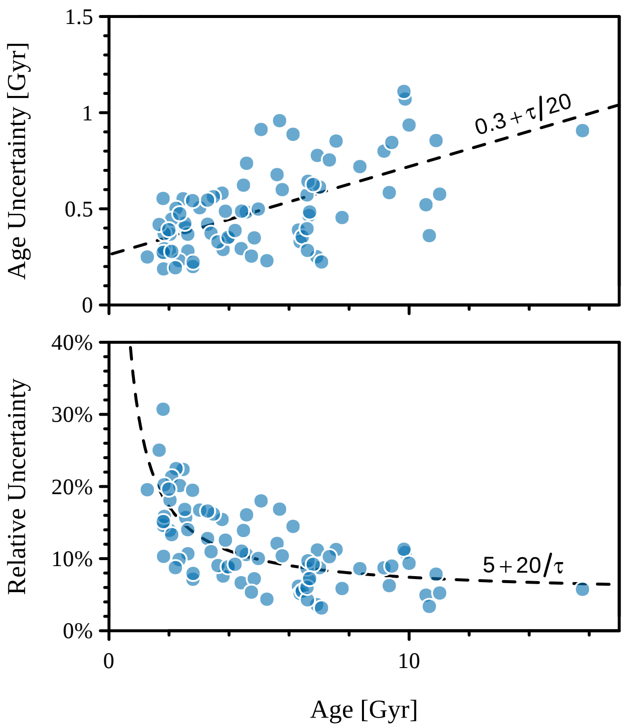

Figure 5. The uncertainties of stellar ages from the literature as a function of age (τ). Top panel: absolute uncertainties; bottom panel: relative uncertainties, in the sense of σ_τ/τ. Trend lines are shown to guide the eye. Older stars have more uncertain ages, in an absolute but not relative sense.

This same procedure may also be used to estimate the stability of the derived exponents. Figure 7 compares the exponents for each of the 80 fits when holding out one star each time. The exponents do not change much between the different fits, which indicates convergence.

It is interesting to try to pinpoint the underlying cause of the discrepancies in the classical scaling relations seen in Figures 2 and 3. As pointed out in the introduction, these relations are derived from

$$\nu_{\max} \propto \nu_{\mathrm{ac}} \propto g/\sqrt{T_{\mathrm{eff}}} \tag{7}$$

$$\Delta\nu \propto \sqrt{\bar{\rho}} \tag{8}$$

where ν_{ac} is the acoustic cut-off frequency of the star, g is its surface gravity and $\bar{\rho}$ is its mean density. Figure 8 compares the left and right sides for both of these relations. While the $\Delta\nu$ scaling relation holds very well (generally within about 1%), the $\nu_{\max}$ scaling relation has much larger scatter. This coincides with other recent findings that the classical $\nu_{\max}$ scaling relation should have additional dependencies (Jiménez et al. 2015; Viani et al. 2017). A more accurate $\nu_{\max}$ relation can easily be inferred from the values given in Table 1.

6 DISCUSSION & CONCLUSIONS

In this paper, I formulated new scaling relations for estimating the mass, radius, and age of solar-type stars. I calibrated the free parameters of these relations using 80 well-studied stars whose ages have been previously determined using detailed fitting to theoretical stellar models. The values for the fitted relations are given in Table 1. I used cross-validation to

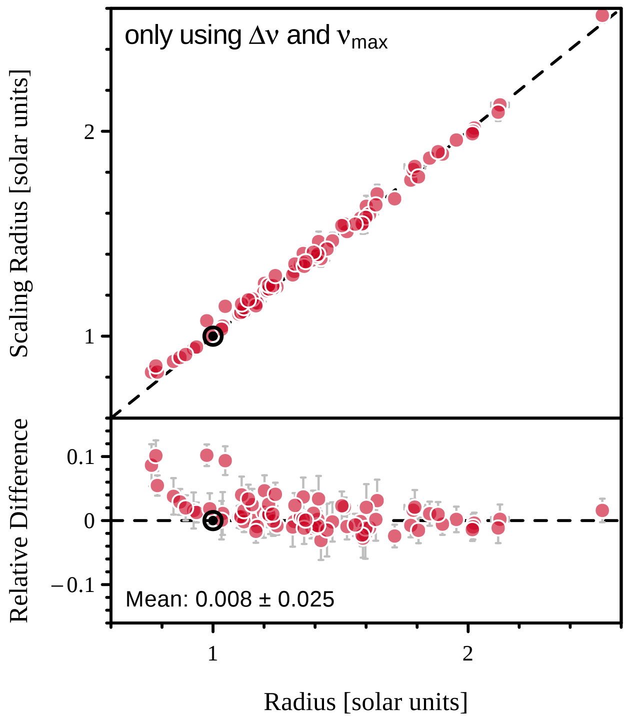

Figure 6. A purely seismic scaling relation for stellar radius (compare with the classical radius scaling relation in Figure 3).

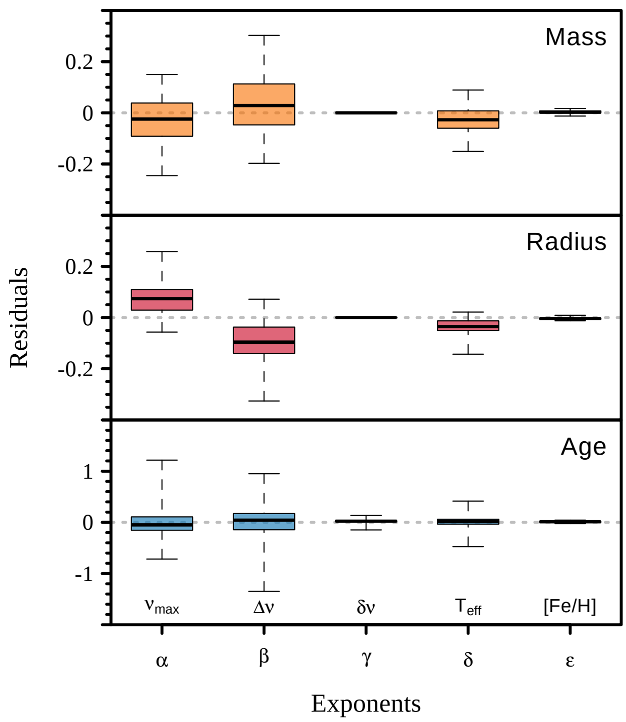

Figure 7. Boxplots comparing the estimated exponents across the 80 cross-validation fits to the values given in Table 1. The middle line shows the median of the residuals, the box shows the interquartile range, and the whiskers extend to the farthest points. The zero line indicates no difference. Note the differences in scale.

gauge the stability of the exponents of the relations, and also to assess the accuracy of the relations. When compared with the classical scaling relations, the new relations were found to be better at reproducing the literature values of stellar mass and radius. The scaling age relation was also found to have a typical precision that is similar to age determination methods which make reference to a grid of theoretical models. For easy use, source code for these relations can be found in the Appendix.

A few points of discussion are in order. These relations have been fit to literature values of age, mass, and radius, which themselves were determined via fits to a grid of theoretical stellar models. Therefore, these relations are model-dependent. It follows that errors in the literature values may affect the accuracy of these relations. Several sources of systematic errors may exist in the theoretical models used to estimate stellar ages, such as unpropagated uncertainties in nuclear reaction rates or atmospheric abundances. When stellar models are inevitably improved, the stars should be fit again, and these relations re-calibrated.

That being said, the literature values used to calibrate these relations have been found to be in good agreement with external constraints, such as *Gaia* radii and luminosities, interferometry, and also with other modeling efforts which are based on different theoretical models (Bellinger et al. 2016, 2018). Furthermore, a large effort was made in the estimation of the stellar parameters to propagate sources of uncertainty stemming from unknown model physics inputs, such as diffusion and overshooting.

One might be tempted to calibrate these relations with stellar models directly, circumventing the need to use real stars whose ages have been fit to said models. However, it must be kept in mind that theoretical values of ν_{max} are generally computed using the scaling relation. Furthermore, theoretical calculations of the large and small frequency separation are affected by the asteroseismic surface term, giving rise to systematic discrepancies between theory and observation. The ages used here were instead determined using asteroseismic frequency ratios, which are insensitive to surface effects (Roxburgh & Vorontsov 2003; Otí Floranes et al. 2005), but more difficult for observers to measure. This approach allows us the convenience of using the observed large and small frequency separations and ν_{max}.

It will be interesting in a future work to calibrate similar relations to more evolved stars such as red giants, where the small frequency separation is no longer a useful diagnostic of stellar age.

ACKNOWLEDGEMENTS

The author thanks Hans Kjeldsen and Jørgen Christensen-Dalsgaard for interesting and useful discussions. Funding for the Stellar Astrophysics Centre is provided by The Danish National Research Foundation (Grant agreement no.: DNRF106). The numerical results presented in this work were obtained at the Centre for Scientific Computing, Aarhus[1].

[1] http://phys.au.dk/forskning/cscaa/

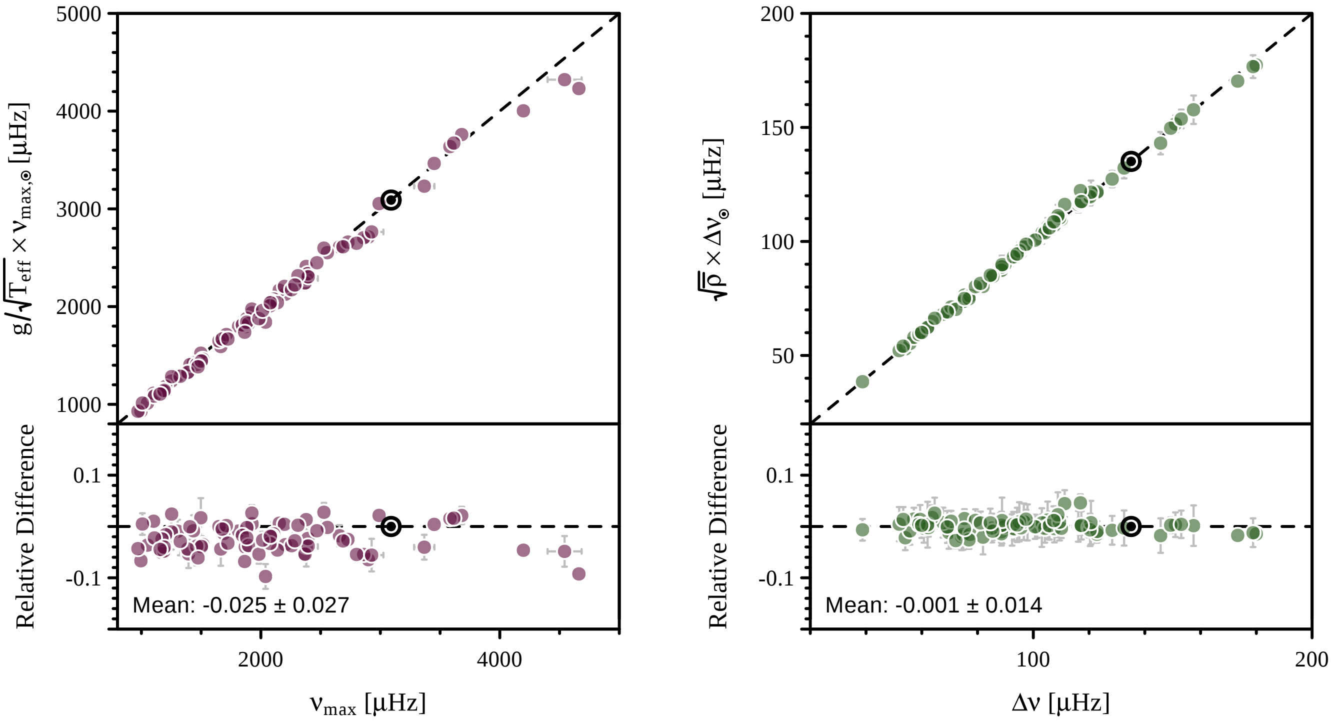

Figure 8. The ν_{max} (left, Equation 7) and $\Delta\nu$ (right, Equation 8) scaling relations. All quantities have been computed in solar units. The surface gravities and mean densities are derived from the literature values of mass and radius; all other quantities are observed.

REFERENCES

Aerts C., Christensen-Dalsgaard J., Kurtz D. W., 2010, Asteroseismology. Springer Science

Angelou G. C., Bellinger E. P., Hekker S., Basu S., 2017, ApJ, 839, 116

Basu S., Chaplin W. J., 2017, Asteroseismic Data Analysis. Princeton

Bellinger E. P., Angelou G. C., Hekker S., Basu S., Ball W. H., Guggenberger E., 2016, ApJ, 830, 31

Bellinger E. P., Basu S., Hekker S., Ball W. H., 2017, ApJ, 851, 80

Bellinger E. P., Hekker S., Angelou G. C., Stokholm A., Basu S., 2018, A&A, 830, 31

Bergemann M., Serenelli A., 2014, Solar Abundance Problem. Springer, pp 245–258, doi:10.1007/978-3-319-06956-2_21

Bland-Hawthorn J., Gerhard O., 2016, ARA&A, 54, 529

Bonanno A., Fröhlich H. E., 2015, A&A, 580, A130

Borucki W. J., et al., 2010, Science, 327, 977

Brown T. M., Gilliland R. L., Noyes R. W., Ramsey L. W., 1991, ApJ, 368, 599

Brown T. M., Christensen-Dalsgaard J., Weibel-Mihalas B., Gilliland R. L., 1994, ApJ, 427, 1013

Christensen-Dalsgaard J., 1984, in Mangeney A., Praderie F., eds, Space Research in Stellar Activity and Variability. p. 11

Davies G. R., Broomhall A. M., Chaplin W. J., Elsworth Y., Hale S. J., 2014, MNRAS, 439, 2025

Davies G. R., et al., 2016, MNRAS, 456, 2183

Foreman-Mackey D., 2016, JOSS, 24

Foreman-Mackey D., Hogg D. W., Lang D., Goodman J., 2013, PASP, 125, 306

Fukunaga K., Hostetler L., 1975, IEEE Transactions on Information Theory, 21, 32

Gaulme P., et al., 2016, ApJ, 832, 121

Guggenberger E., Hekker S., Basu S., Bellinger E., 2016, MNRAS, 460, 4277

Guggenberger E., Hekker S., Angelou G. C., Basu S., Bellinger E. P., 2017, MNRAS, 470, 2069

Huber D., et al., 2011, ApJ, 743, 143

Jiménez A., García R. A., Pérez Hernández F., Mathur S., 2015, A&A, 583, A74

Kjeldsen H., Bedding T. R., 1995, A&A, 293, 87

Lebreton Y., Goupil M. J., 2014, A&A, 569, A21

Lund M. N., et al., 2017, ApJ, 835, 172

Lundkvist M., Kjeldsen H., Silva Aguirre V., 2014, A&A, 566, A82

Metcalfe T. S., et al., 2014, ApJS, 214, 27

Nissen P. E., Silva Aguirre V., Christensen-Dalsgaard J., Collet R., Grundahl F., Slumstrup D., 2017, A&A, 608, A112

Otí Floranes H., Christensen-Dalsgaard J., Thompson M. J., 2005, MNRAS, 356, 671

Pedregosa F., et al., 2011, JMLR, 12, 2825

Pont F., Eyer L., 2004, MNRAS, 351, 487

Prša A., et al., 2016, AJ, 152, 41

Rendle B. M., et al., 2019, MNRAS, 484, 771

Roxburgh I. W., Vorontsov S. V., 2003, A&A, 411, 215

Silva Aguirre V., et al., 2015, MNRAS, 452, 2127

Silva Aguirre V., et al., 2017, ApJ, 835, 173

Silva Aguirre V., et al., 2018, MNRAS, 475, 5487

Soderblom D. R., 2010, ARA&A, 48, 581

Stello D., Chaplin W. J., Basu S., Elsworth Y., Bedding T. R., 2009, MNRAS, 400, L80

Themeßl N., et al., 2018, MNRAS, 478, 4669

Ulrich R. K., 1986, ApJ, 306, L37

Viani L. S., Basu S., Chaplin W. J., Davies G. R., Elsworth Y., 2017, ApJ, 843, 11

White T. R., Bedding T. R., Stello D., Christensen-Dalsgaard J., Huber D., Kjeldsen H., 2011, ApJ, 743, 161

Winn J. N., Fabrycky D. C., 2015, ARA&A, 53, 409

APPENDIX A: SOURCE CODE

For the sake of convenience, here I provide source code in Python 3 to make use of these new scaling relations.

scaling.py

```
from math import e
from uncertainties import ufloat

# Enter some data, for example a solar twin
# ufloat holds a measurement and its uncertainty
nu_max   = ufloat(3090,  3090  * 0.01)  # muHz, with 1%   uncertainty
Delta_nu = ufloat(135.1, 135.1 * 0.001) # muHz, with 0.1% uncertainty
delta_nu = ufloat(8.957, 8.957 * 0.04)  # muHz, with 4%   uncertainty
Teff     = ufloat(5772,  5772  * 0.01)  # K,    with 1%   uncertainty
Fe_H     = ufloat(0,             0.1)   # dex,  0.1 dex   uncertainty

# Take the powers from Table 1, here given with more precision
# P        = [      alpha,        beta,       gamma,      delta,      epsilon]
P_mass     = [ 0.97531880, -1.43472745,  0,           1.21647950,  0.27014278]
P_radius   = [ 0.30490057, -1.12949647,  0,           0.31236570,  0.10024562]
P_R_seis   = [ 0.88364851, -1.85899352,  0,           0,           0         ]
P_age      = [-6.55598425,  9.05883854, -1.29229053, -4.24528340, -0.42594767]

# Apply the scaling relation
def scaling(nu_max, Delta_nu, delta_nu, Teff, exp_Fe_H, P=P_age,
            nu_max_Sun = ufloat(3090,  30),     # muHz
          Delta_nu_Sun = ufloat(135.1, 0.1),    # muHz
          delta_nu_Sun = ufloat(8.957, 0.059), # muHz
              Teff_Sun = ufloat(5772,  0.8),    # K
              Fe_H_Sun = ufloat(0,     0)):     # dex

    alpha, beta, gamma, delta, epsilon = P

    # Equation 5
    return ((nu_max   /   nu_max_Sun) ** alpha *
            (Delta_nu / Delta_nu_Sun) ** beta  *
            (delta_nu / delta_nu_Sun) ** gamma *
            (Teff     /     Teff_Sun) ** delta *
            (e**Fe_H  /  e**Fe_H_Sun) ** epsilon)

scaling_mass   = scaling(nu_max, Delta_nu, delta_nu, Teff, Fe_H, P=P_mass)
scaling_radius = scaling(nu_max, Delta_nu, delta_nu, Teff, Fe_H, P=P_radius)
scaling_age    = scaling(nu_max, Delta_nu, delta_nu, Teff, Fe_H, P=P_age) * \
    ufloat(4.569, 0.006)

print('Mass:',   scaling_mass,   '[solar units]')
print('Radius:', scaling_radius, '[solar units]')
print('Age:', '{:.2u}'.format(scaling_age), '[Gyr]')
```

```
$ python3 scaling.py
Mass: 1.000+/-0.033 [solar units]
Radius: 1.000+/-0.011 [solar units]
Age: 4.57+/-0.56 [Gyr]
```

This paper has been typeset from a TEX/LATEX file prepared by the author.