

# Pulsating hydrogen-deficient white dwarfs and pre-white dwarfs observed with *TESS*

## VI. Asteroseismology of the GW Vir-type central star of the Planetary Nebula NGC 246

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### ABSTRACT

**Context.** Significant advances have been achieved through the latest improvements in the photometric observations accomplished by the recent space missions, which substantially boost the study of pulsating stars via asteroseismology. The *TESS* mission has already proven to be of particular relevance for pulsating white dwarf and pre-white dwarf stars.

**Aims.** We report a detailed asteroseismic analysis of the pulsating PG 1159 star NGC 246 (TIC 3905338), which is the central star of the planetary nebula NGC 246, based on high-precision photometric data gathered by the *TESS* space mission.

**Methods.** We reduced *TESS* observations of NGC 246 and performed a detailed asteroseismic analysis using fully evolutionary PG 1159 models computed accounting for the complete prior evolution of their progenitors. We constrained the mass of this star by comparing the measured mean period spacing with the average of the computed period spacings of the models, and we also employed the observed individual periods to search for a seismic stellar model.

**Results.** We extracted a total of 17 periodicities from the *TESS* light curves from the two sectors where NGC 246 was observed. All the oscillation frequencies are associated with *g*-mode pulsations, with periods spanning from  $\sim 1460$  to  $\sim 1823$  s. We found a constant period spacing of  $\Delta\Pi = 12.9$  s, which allowed us to deduce that the stellar mass is larger than  $\sim 0.87 M_{\odot}$  if the period spacing is assumed to be associated with  $\ell = 1$  modes, and that the stellar mass is  $\sim 0.568 M_{\odot}$  if it is associated with  $\ell = 2$  modes. Although we were not able to find a unique asteroseismic model for this star, the period-to-period fit analyses suggest a high-stellar mass ( $\gtrsim 0.74 M_{\odot}$ ) when the observed periods are associated with modes with  $\ell = 1$  only, and both a high ( $\gtrsim 0.74 M_{\odot}$ ) and intermediate ( $\sim 0.57 M_{\odot}$ ) stellar mass when the observed periods are associated with modes with a mixture of  $\ell = 1, 2$ . We also assessed the distances corresponding to the mentioned possible seismic solutions and found that the asteroseismic distance for a model with  $M_{\star} = 0.565 M_{\odot}$ , characterized by approximately the same effective temperature of NGC 246 ( $\sim 150\,000$  K), is compatible with the astrometric distance predicted by the precise *Gaia* measurements.

**Conclusions.**

**Key words.** stars: individual (NGC 246) — asteroseismology — white dwarfs — stars: evolution — stars: interiors

## 1. Introduction

GW Vir stars are pulsating PG 1159 stars, that is, pulsating hot hydrogen(H)-deficient white dwarfs (WDs) and pre-WDs with surface layers rich in helium (He), carbon (C), and oxygen (O) (Werner & Herwig 2006; Werner et al. 2014; Werner & Rauch 2015; Córscico et al. 2019; Sowicka et al. 2023). The class of GW Vir stars is often separated into the so-called variable planetary nebula nuclei (PNNVs) that are still surrounded by a nebula, and

DOVs<sup>1</sup>, objects that lack a nebula. Among GW Vir stars also are the pulsating Wolf-Rayet central stars of a planetary nebula ([WC]) and early-[WC] ([WCE]) stars, since they share the pulsation properties of pulsating PG 1159 stars (Quirion et al. 2007). PG 1159 stars are thought to be the evolutionary link between post-asymptotic giant branch (AGB) stars and most of the H-deficient WDs (Althaus et al. 2005, 2010; Werner & Her-

<sup>1</sup> Even though white dwarf stars of spectral type DO do not pulsate, the term "DOVs" has historically been used as a variable star designation for GW Vir pulsators without a nebula.

wig 2006; Sowicka et al. 2021). The origin of these stars is likely to be in the context of a single-star evolution in a born-again episode induced by a delayed (late or very late) post-AGB He thermal pulse (Herwig 2001; Blöcker 2001; Althaus et al. 2005; Miller Bertolami & Althaus 2006), or binary star evolution in a binary WD merger (Miller Bertolami et al. 2022; Werner et al. 2022). GW Vir stars exhibit multiperiodic brightness variations with periods ranging from 300 to 6000 s associated with low-degree ( $\ell \leq 2$ ) nonradial gravity( $g$ )-mode pulsations excited by the  $\kappa$  mechanism due to partial ionization of C and O in the outer layers (Starrfield et al. 1983, 1984; Stanghellini et al. 1991; Bradley & Dziembowski 1996; Saio 1996; Gautschy 1997; Gautschy et al. 2005; Córscico et al. 2006; Quirion et al. 2007). DOVs and PNNVs share the same mode driving mechanism, regardless of the absence or presence of a nebula.

Remarkably, significant observational efforts have been made in the last years to study pulsating WDs and pre-WDs, among them, the GW Vir stars (Córscico et al. 2019). From dedicated ground-based observations (see, e.g., the works from Vauclair et al. 2002; Fu et al. 2007; Costa et al. 2008; Kepler et al. 2014; Sowicka et al. 2023), to the unprecedented high-quality data brought about by the advent of space missions, asteroseismology of GW Vir stars has been significantly encouraged. In the context of space observations, the *Kepler* extended mission (*K2*, Howell et al. 2014) and its successor, the Transiting Exoplanet Survey Satellite (*TESS*, Ricker et al. 2015), have promoted this field even further. The high-sensitivity and continuous observation by *TESS* of  $\sim 85\%$  of the sky have made the discovery of an enormous number of pulsating stars possible (see, e.g., Kurtz 2022), making valuable contributions in the field of WDs and pre-WDs (Córscico 2022). Particularly relevant for the present paper are the works by Córscico et al. (2021), dedicated to the analysis of a set of already known GW Vir stars, and Uzundag et al. (2021, 2022), where a total of four new members of this type of stars were presented. In addition, combining *Kepler/K2* and *TESS* observations, Oliveira da Rosa et al. (2022) have presented a thorough asteroseismic analysis applied to PG 1159-035, the prototype of GW Vir stars (McGraw et al. 1979). In these works, detailed asteroseismic studies have been carried out, leading to the characterization and determination of the fundamental parameters of these stars.

In the present paper, we report new *TESS* observations of NGC 246 (TIC 3905338, according to its *TESS* designation), an already-known GW Vir star that is the central star of the planetary nebula NGC 246<sup>2</sup>, and carry out a detailed asteroseismic analysis on the basis of the full PG 1159 evolutionary models of Althaus et al. (2005) and Miller Bertolami & Althaus (2006). The NGC 246 planetary nebula has a slightly elliptical shape, and its central star (HIP 3678) is a hierarchical triple stellar system (Adam & Mugrauer 2014), the only confirmed such system in a planetary nebula. The primary, NGC 246 (a.k.a. HIP 3678 A) is a PG 1159 star with  $T_{\text{eff}} = 150\,000$  K and  $\log(g) = 5.7$  cgs, according to the non-LTE model atmosphere analysis by Rauch & Werner (1997), and  $M_{\star} \sim 0.75 M_{\odot}$  (Werner & Herwig 2006; Miller Bertolami & Althaus 2006). Its orbital companions (HIP 3678 B & C) are likely low-mass main sequence stars of spectral type K and M (Adam & Mugrauer 2014). The location of NGC 246 in the  $\log T_{\text{eff}} - \log g$  diagram is displayed in Fig. 1, along with other GW Vir stars. A more recent analysis carried out by Löbbling (2018, 2020), employing state-of-the-art

non-LTE atmosphere models, considering opacities of 17 chemical species, including iron-group elements, and employing new spectroscopic observations, confirmed the previous results, but with fairly tight error estimates:  $T_{\text{eff}} = 150\,000 \pm 10\,000$  K and  $\log g = 5.7 \pm 0.1$ . A comparison with the evolutionary tracks of Miller Bertolami & Althaus (2006) results in a stellar mass of  $0.74^{+0.19}_{-0.23} M_{\odot}$  (Löbbling 2018, 2020).

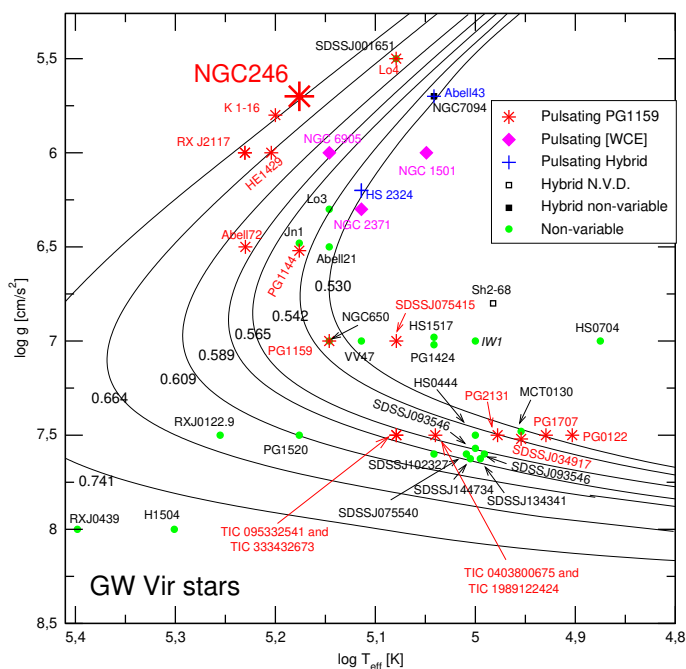
In the first literature notes from Landolt (1983) and Grauer et al. (1987), this star was listed as a non-variable star. Only a decade later, the observations of Ciardullo & Bond (1996) carried out on three nights showed its low-amplitude pulsations. The first two (consecutive) nights revealed significant peaks at 683 and 648  $\mu\text{Hz}$  in the Fourier spectrum, while the last observation, nine months later, showed a different peak at 540  $\mu\text{Hz}$ . A more extensive study conducted by González Pérez et al. (2003) (three short runs carried out in the years 2000-2001) showed a complex and highly variable short-scale photometric behavior. In the 2000 run, the first night (2.2 h long) showed a peak around 550  $\mu\text{Hz}$ , while the second (3 h, three days later) showed a lower amplitude peak in the same region and, also, a stronger peak at 685  $\mu\text{Hz}$ . Observations from over a year later (more than 4 h long) showed a peak at  $\sim 450$   $\mu\text{Hz}$  and additional possible peaks located around 400 and 500  $\mu\text{Hz}$ . The last run was carried out two months later and, because of its poorer quality, it only showed low-amplitude peaks at the same region as in the previous run. Moreover, the authors claimed possible features in the Fourier spectra that may be related to interactions with a close companion (for instance, alleged harmonics of the 450  $\mu\text{Hz}$  peak). All these observations, carried out over more than two decades, only proved that in order to reveal the details of the complex nature of the pulsations of the central star in NGC 246, extensive monitoring on a longer time base was very necessary. With a lack of a dedicated Whole Earth Telescope (WET) campaign (Nather et al. 1990), as already suggested by González Pérez et al. (2003), the *TESS* observations finally give us the exciting opportunity to study the complex behavior of the GW Vir star in NGC 246 in detail. A preliminary analysis of this star has been previously presented by Aller et al. (2020).

The present work is the sixth in the series of papers dedicated to studying pulsating H-deficient WDs and pre-WDs observed with *TESS*. The first paper was focused on the analysis of six already known GW Vir stars (Córscico et al. 2021), and the second one, on the discovery of two new stars of this type (Uzundag et al. 2021). In the third part, Córscico et al. (2022c) analyzed the DBV star<sup>3</sup> GD 358, while in the fourth (Uzundag et al. 2022), the discovery of two additional GW Vir stars was reported. In the fifth paper (Córscico et al. 2022b), the authors presented the discovery of two new DBV pulsators and the analysis of three already-known pulsating WD stars of this type.

This paper is organized as follows. In Sect. 2, we provide details of the observations and reduction of the photometric *TESS* data and also, on the frequency analysis. In Sect. 3, we start by giving a brief summary of the evolutionary models and numerical tools employed for our theoretical analyses. Next, we focus on the asteroseismic modeling of NGC 246, including the search for a possible mean period spacing within the period spectra (Sect. 3.1), the derivation of the stellar mass using the inferred period spacing (Sect. 3.2), and the search for a period-to-period fit with the aim of finding an asteroseismic model that best rep-

<sup>2</sup> Although NGC 246 is the name of the planetary nebula, throughout this work we use this designation to refer to the central star, for simplicity.

<sup>3</sup> DBVs are He-rich atmosphere pulsating WD stars with effective temperatures and surface gravities in the ranges  $22\,400 \lesssim T_{\text{eff}} \lesssim 32\,000$  K and  $7.5 \lesssim \log g \lesssim 8.3$ , respectively (Córscico et al. 2019).



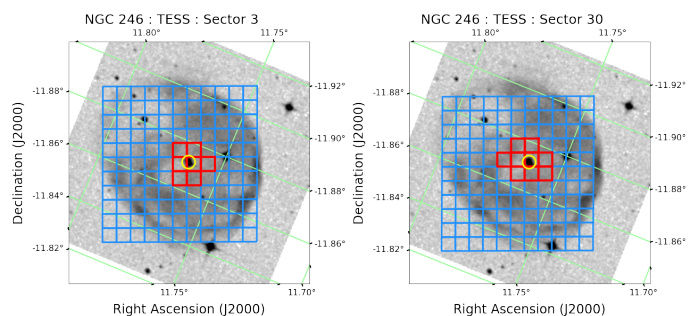
**Fig. 1.** The already known variable and non-variable PG 1159 stars and variable [WCE] stars in the  $\log T_{\text{eff}} - \log g$  diagram. Thin solid curves show the post-born again evolutionary tracks from Miller Bertolami & Althaus (2006) for different stellar masses. "N.V.D." stands for PG 1159 stars with no variability data. "Hybrid" refers to PG 1159 stars exhibiting H in their atmospheres. The location of the GW Vir star in NGC 246 is emphasized with a large red star symbol. This star is characterized by  $T_{\text{eff}} = 150\,000$  K and  $\log g = 5.7$ .

resents the pulsations of the target star (Sect. 3.3). Finally, in Sect. 4 we summarize our main findings.

## 2. Observations and data reduction

NGC 246 was observed by *TESS* in Sector 3 between September 20 and October 18, 2018, and in Sector 30 between September 22 and October 21, 2020. In Sector 3, this star was observed during 20.27 days with the short cadence (SC, 120 s) mode, with a 1.19-day gap between the orbits, which is shorter than the 27-day nominal duration of one-sector observations, while in Sector 30 it was observed for 25.92 days with a 1.07-day gap in both SC and ultra-short cadence mode (USC, 20 s). There is a 709.67-day data gap between the two sectors.

We first investigated the target-pixel files (TPFs) from *TESS* to assess the potential contamination from the nearby objects within the *TESS* aperture by using different aperture shapes, sizes, and systematics corrections. We downloaded TPFs for Sectors 3 and 30 in LIGHTKURVE (Lightkurve Collaboration et al. 2018) for further processing. Then, we created custom apertures and used them to extract raw light curves from the TPFs. TPFs are already processed by the Science Processing Operations Center (SPOC) pipeline (Jenkins et al. 2016) that removes backgrounds, but it is still possible to remove common trends in TPF pixels. We found that Pixel Level Decorrelation (PLD) (Deming et al. 2015) within LIGHTKURVE gave the best results compared to the Pre-search Data Conditioning Simple Aperture Photometry (PDCSAP) flux. PLD is a special case of a Regression Corrector that uses linear regression to remove systematic noise from light curves based on a design matrix created using only background pixels surrounding the target aperture. A noise



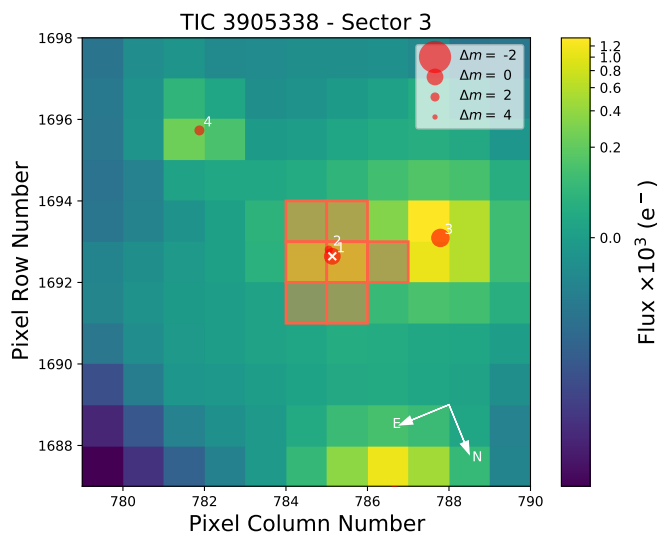
**Fig. 2.** *TESS* Sectors 3 and 30 pipeline aperture overlaid on DSS2 Red sky image. The target is marked with yellow circles. Blue squares show the TPF overlay and red squares show the TPF aperture overlay for each Sector. Figure produced using mkpy3 by Kenneth Mighell (<https://github.com/KenMighell/mkpy3>).

model created using this information from nearby pixels is then subtracted from the uncorrected light curve.

In Fig. 2, we show the pipeline apertures used to extract photometry in Sectors 3 and 30, overlaid on the DSS2 Red image. It is evident that nebular contamination is small but inevitable, and it varies between the two sectors because of the different sizes of the apertures. The on-sky orientation of the field is almost the same (there is only a small,  $\sim 1$  pix shift), therefore the use of an aperture of the same shape (but slightly shifted) would be justified and perhaps more appropriate in this case.

Figure 3 shows the target ( $T_{\text{mag}} = 12.31$ ) with other stars from the *TESS* input catalog (Stassun et al. 2019). There are two main sources of contamination: the  $T_{\text{mag}} = 13.70$  star close to the target position which is the wide binary member of the NGC 246 triple system (HIP 3678 B), and the  $T_{\text{mag}} = 11.14$  star, TYC 5272-1854-1, to the right. The *TESS* header contains two parameters for crowding correction: CROWDSAP - the crowding metric, defined as the ratio of the target flux to the total flux in the optimal aperture, and FLFRCSAP - the flux fraction, which reflects what fraction of the Pixel Response Function of the target is outside the optimal aperture (missing flux). In the case of NGC 246, the CROWDSAP value is 0.72 for Sector 3 and 0.64 for Sector 30, while FLFRCSAP is 0.72 and 0.78, respectively. It means that using solely the CROWDSAP, only 72 and 64% of the total flux originally measured in the aperture originates from the target in Sectors 3 and 30, respectively. The possible influence on the variability of NGC 246 by the 13.70 mag star is hard to assess with only *TESS* data at hand, therefore we checked whether the 11.14 star shows any variability in *TESS* data. We varied aperture sizes and shapes for that target and concluded that there was no variability from that star that could influence our analysis of NGC 246. With the ground-based data from González Pérez et al. (2006) reduced anew, using only a short run where it was possible to resolve the stars, we were not able to verify the lack of variability in the 13.70 mag star on time scales relevant to our analysis. Since this star, HIP 3678 B, is an early-to-mid K dwarf star (Adam & Mugrauer 2014), it is not expected to exhibit variability that we could confuse with pulsations of the white dwarf primary. Our new analysis of this data set only confirms the findings reported by the authors that the Fourier spectrum was variable on a nightly basis.

For the analysis presented in the remainder of the paper, we extracted times in Barycentric corrected Julian days ("BJD-245700") and fluxes already corrected for contamination by the pipeline ("PDCSAP FLUX") from the FITS files using LIGHTKURVE. The fluxes were normalized and transformed to am-



**Fig. 3.** TESS Target Pixel File (TPF) with the position of NGC 246 marked with a white “X”. The other sources in the field from Gaia DR2 are marked with red circles scaled with magnitudes. The aperture mask used by the PDC pipeline is shown with red squares.

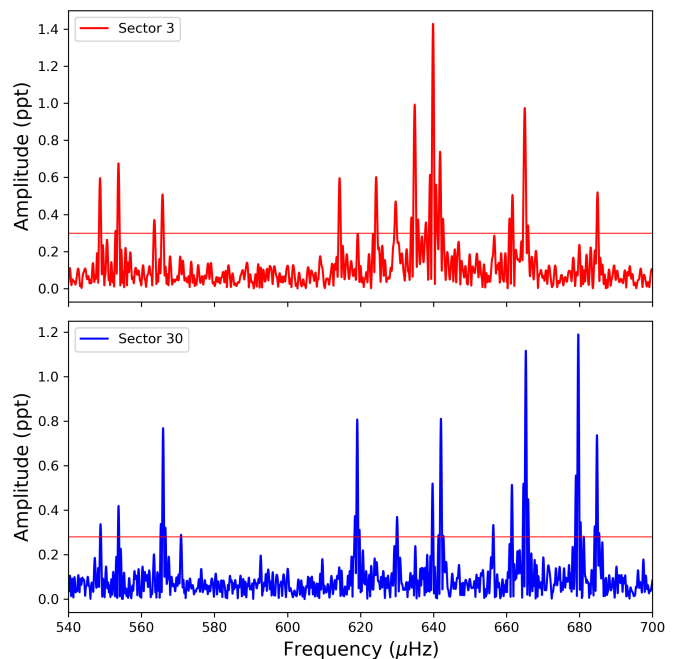
plitudes in parts-per-thousand (ppt) units. To remove the outliers that appear above 5 times the median of intensities, the data were finally sigma-clipped based on  $5\sigma$ .

### 2.1. Frequency analysis

In order to analyze the periodicities in the data and determine the frequency of each pulsation mode, along with their amplitude and phase, Fourier transforms (FT) of the light curves were computed. We choose a 0.1 percent false alarm probability (FAP) detection threshold, which means that if the amplitude exceeds this threshold, there is a 0.1 percent chance that it is merely the result of noise fluctuations. Following the procedure outlined in Kepler (1993), we determined the 0.1% FAP threshold. We performed a nonlinear least square (NLLS) fit in the form of  $A_i \sin(\omega_i t + \phi_i)$ , with  $\omega = 2\pi/P$ , where  $P$  is the period. We identified the frequency (period), phase, and amplitude values for each periodicity in this way. We prewhitened the light curves using the NLLS fit parameters until, excepting unresolved peaks, there was no longer any significant signal in the FT of both datasets above the significance level of 0.1% FAP. All prewhitened frequencies for NGC 246 are given in Table 1, where we present frequencies (periods) and amplitudes, along with their corresponding errors and the S/N ratio. The average noise level of Sector 3 corresponds to 0.06 ppt and the significance level of 0.1% FAP, to 0.29 ppt. We extracted 15 frequencies above 0.29 ppt from the light curve, as shown in the top panel of Fig. 4. All the pulsational frequencies are located between  $548 \mu\text{Hz}$  (1822 s) and  $684 \mu\text{Hz}$  (1459 s). For Sector 30, the average noise level corresponds to 0.055 ppt. In total, we detected 12 pulsational frequencies above the detection threshold of 0.26 ppt. This case is shown in the bottom panel of Fig. 4. In Table 1, we reported the final best-fit values for the frequencies (periods) and amplitudes of the individual sinusoids in our model.

### 2.2. Possible rotational multiplets?

As we will see, identification of the value of  $\ell$  is a critical component of an asteroseismic analysis. One property of nonradial pul-



**Fig. 4.** Fourier transform of Sector 3 (upper panel) and Sector 30 (lower panel) of NGC 246. The horizontal red line indicates the 0.1% FAP level.

sations in compact stars – the effects of rotation – often provides critical data to identify the value of  $\ell$  (and  $m$ ). For a nonrotating star, the pulsation frequencies for modes with the same value of  $k$  are the same for all values of  $\ell$  and  $m$ . Rotation, however, can lift that degeneracy, splitting modes with different sets of  $(\ell, m)$  by amounts proportional to the rotation frequency  $\Omega$ . As long as the rotation frequency is significantly smaller than the pulsation frequency (i.e. to first order in  $\Omega/\sigma_k$ ), we have:

$$\sigma_{k,\ell,m} = \sigma_k + m\Omega(1 - C_{k,\ell}) \quad (1)$$

where  $\sigma_k$  is the oscillation (angular) frequency in the nonrotating case, and  $C_{k,\ell}$  is a function (integrated through the star) of the stellar structure and the eigenfunctions for the mode (for details see e.g. Aerts et al. 2010).

Thus, rotation can produce a triplet of  $\ell = 1$  modes, with a central peak ( $m = 0$ ) flanked by peaks separated on either side by  $\Omega*(1 - C_{k,\ell})$ . Higher  $\ell$  modes can be split into multiplets with  $2\ell + 1$  components, split equally by the same factor (though the value of  $C_{k,\ell}$  depends on  $\ell$ ).

For  $g$ -modes in compact stars, where the horizontal displacement is much larger than the radial displacement, the value of  $C_{k,\ell}$  is closely approximated by (Winget et al. 1991):

$$C_{k,\ell} = \frac{1}{\ell(\ell + 1)}. \quad (2)$$

So for  $\ell = 1$  modes, the (angular) frequency spacing between peaks in the FT is  $\Omega/2$ , while for  $\ell = 2$  modes we would (ideally) see a quintuplet of modes, split by  $\Omega \times 5/6$ . The rotational splitting of all modes (all values of  $k$ ) of a given  $\ell$  should be identical unless there is latitudinal differential rotation. If both  $\ell = 1$  and  $\ell = 2$  modes are excited in the star, the values of their rotational splitting should be related by exactly  $\frac{\Delta f_{\ell=1}}{\Delta f_{\ell=2}} = 0.6$ .

If the rotational period is not longer than the duration of the observation, this  $2\ell + 1$  configuration can be resolved using high-precision photometry. Rotational multiplets have been detected

for all types of pulsating WDs, including GW Vir, DBV, and DAV<sup>4</sup> stars, and the derived rotation periods range from 1 hour to a few days (Hermes et al. 2017; Córscico et al. 2019, 2022a; Uzundag et al. 2022).

For a rotation period of 1 or 2 days, the  $\ell = 1$  splitting corresponds to 5.79 or 2.89  $\mu\text{Hz}$ , respectively. The  $\ell = 2$  splitting is larger (see Equation 2) by a factor of 5/3. In most of the cases cited above, the frequency splitting between peaks in a given  $(k, \ell)$  multiplet are significantly smaller than the spacing between modes of different  $k$ . But for pulsating PN central stars, which have periods that are significantly longer, the separation in frequency between modes of different  $k$  are smaller. For the long periods as those detected in NGC 246, the period spacing between consecutive  $m=0$  modes results in a small frequency spacing which can be confused with the rotational splitting, so care must be exercised to separate the two effects (similar issues arose in the study of the central star of NGC 1501 by Bond et al. 1996).

With this theoretical background, we attempted a search for frequency splittings within the detected frequencies. First, we searched for pairs of peaks with a similar frequency separation. Frequencies  $f_{10,S03}$  and  $f_{11,S03}$  differ by  $641.829 - 639.872 = 1.957 \mu\text{Hz}$ , while frequencies  $f_{14,S03}$  and  $f_{15,S03}$  by  $664.997 - 661.711 = 3.286 \mu\text{Hz}$ . The relation between those separations is almost exactly 0.6, therefore we might suspect that  $f_{10,S03}$  and  $f_{11,S03}$  could be  $\ell = 1$  modes, whereas  $f_{14,S03}$  and  $f_{15,S03}$  could be  $\ell = 2$  modes, both cases with missing components. Another pair of  $\ell = 1$  modes could be  $f_{3,S03}$  and  $f_{4,S03}$ :  $565.800 - 563.500 = 2.300 \mu\text{Hz}$ . On the other hand, the frequency splitting might be larger, as we found four sequences split by approximately  $5 \mu\text{Hz}$  (it could even be a sequence of five modes,  $f_6 - f_{10}$ ). Those are:  $f_{1,S03}$  and  $f_{2,S03}$ :  $553.725 - 548.646 = 5.079 \mu\text{Hz}$ ,  $f_{6,S03}$  and  $f_{7,S03}$ :  $624.294 - 619.230 = 5.064 \mu\text{Hz}$ ,  $f_{9,S03}$  and  $f_{10,S03}$ :  $639.872 - 634.850 = 5.022 \mu\text{Hz}$ , and  $f_{13,S30}$  and  $f_{14,S30}$ :  $661.506 - 656.412 = 5.094 \mu\text{Hz}$ . If those modes were  $\ell = 1$  we could expect the  $\ell = 2$  modes split by about  $8.4 \mu\text{Hz}$ , otherwise we could expect the  $\ell = 1$  modes split by about  $3 \mu\text{Hz}$ . We did not find any clear sequence with those values other than already mentioned, therefore we explored a possibility of overlapping mode sequences with missing components. For example, if we consider the difference between  $f_{7,S03}$  and  $f_{11,S03}$  of  $641.829 - 624.294 = 17.535 \mu\text{Hz}$ , it could be a triplet of  $\ell = 2$  modes with a missing component in between split by  $8.767 \mu\text{Hz}$  (and missing side components), overlapping in the same region as previously mentioned  $\ell = 1$  modes split by about  $5 \mu\text{Hz}$ . A multitude of options prevents us from a definitive determination of the rotation period in NGC 246 and mode identification based on the observed frequency splitting.

### 3. Asteroseismic analysis

The asteroseismic analysis performed in this work is based on a set of stellar models that take the complete evolution of the PG 1159 progenitor stars into account (Althaus et al. 2005; Miller Bertolami & Althaus 2006, 2007a,b). Post-AGB evolutionary sequences, computed with the LPCODE evolutionary code (Althaus et al. 2005), were followed through the very late thermal pulse (VLTP) episode and the resulting born-again episode that leads to the H-deficient, He-, C-, and O-rich composition expected for PG 1159 stars. The resulting stellar remnants have

<sup>4</sup> DAVs are H-rich atmosphere pulsating WD stars with effective temperatures and surface gravities in the ranges  $10\,400 \lesssim T_{\text{eff}} \lesssim 12\,400 \text{ K}$  and  $7.5 \lesssim \log g \lesssim 9.1$ , respectively (Córscico et al. 2019).

masses of 0.515, 0.530, 0.542, 0.565, 0.589, 0.609, 0.664, 0.741, and 0.870  $M_{\odot}$ . In Fig 1, the evolutionary tracks corresponding to the PG 1159 models employed in this work are shown in the  $\log(T_{\text{eff}}) - \log(g)$  plane. Details about the input physics and evolutionary calculations carried out to obtain the PG 1159 evolutionary sequences employed in this work can be found in Althaus et al. (2005); Miller Bertolami & Althaus (2006, 2007a,b). We computed  $\ell = 1, 2$   $g$ -mode adiabatic pulsation periods in the range 80 - 6000 s with the adiabatic version of the pulsation code LP-PUL (Córscico & Althaus 2006; Córscico et al. 2006). We analyzed roughly 4000 PG 1159 models covering a wide range of effective temperatures ( $4.8 \lesssim \log T_{\text{eff}} \lesssim 5.4$ ), luminosities ( $0 \lesssim \log(L_{\star}/L_{\odot}) \lesssim 4.2$ ), and stellar masses ( $0.515 \leq M_{\star}/M_{\odot} \leq 0.870$ ).

The nonradial  $g$  modes that cause brightness variations in WDs and pre-WDs can be excited in a sequence of consecutive radial orders,  $k$ , for a given value of the harmonic degree,  $\ell$ . In the asymptotic limit of high radial order, the periods of  $g$  modes with consecutive radial orders are approximately evenly separated, being the period spacing dependent on  $\ell$ , according to the expression (Tassoul et al. 1990):

$$\Delta\Pi_{\ell}^a = \Pi_{k+1,\ell} - \Pi_{k,\ell} = \frac{\Pi_0}{\sqrt{\ell(\ell+1)}}, \quad (3)$$

where  $\Pi_0$  is a constant value given by:

$$\Pi_0 = \frac{2\pi^2}{\left[ \int_{r_1}^{r_2} \frac{N}{r} dr \right]} \quad (4)$$

and  $N$  is the Brunt-Väisälä frequency. In the case of chemically homogeneous stellar models, this formula for the asymptotic period spacing ( $\Delta\Pi_{\ell}^a$ ) represents a precise description of the separation of consecutive  $g$ -mode periods for large radial orders. A very relevant property of the  $g$ -mode period spacing of pulsating PG 1159 stars is that it primarily depends on the stellar mass, and only weakly on the luminosity and the He-rich envelope mass (Kawaler 1986, 1987, 1988, 1990; Kawaler & Bradley 1994; Córscico & Althaus 2006). This property, in principle, can be used to derive the mass of a star by measuring the period spacing. For chemically stratified stars, such as PG 1159 stars, the  $g$ -mode period spacings considerably depart from uniformity due to the mechanical resonance called “mode trapping” (Kawaler & Bradley 1994). Therefore, at first glance, it would seem that the observed period separation of a chemically stratified star could not be used to infer its stellar mass. Fortunately, the average value of the period spacings in chemically layered PG 1159 stars still preserves the property of being a function almost exclusively of the stellar mass (Tassoul et al. 1990). Thus, one can still derive the stellar mass of chemically stratified stars from the comparison between the observed mean spacing of periods,  $\Delta\Pi$ , and the average of the period spacings,  $\overline{\Delta\Pi}$ , calculated for sets of PG 1159-star models with different masses and effective temperatures.

Another approach to deriving the stellar mass, which, at the same time, can give information about the internal structure of the pulsating star, is to search for a pulsation model that best matches the *individual* pulsation periods of the star under study. The goodness of the match between the theoretical pulsation periods ( $\Pi_k^T$ ) and the observed individual periods ( $\Pi_i^O$ ) is measured by means of a merit function defined as

**Table 1.** Frequency solution from the light curves of Sectors 3 and 30 of NGC 246 including frequencies, periods, amplitudes (and their uncertainties), and S/N.

ID	Frequency (S03)	Period (S03)	Amplitude (S03)	S/N	Frequency (S30)	Period (S30)	Amplitude (S30)	S/N
	$\mu\text{Hz}$	[sec]	[ppt]		$\mu\text{Hz}$	[sec]	[ppt]	
f <sub>1</sub>	548.646 (27)	1822.67 (9)	0.59 (7)	9.2	548.774 (39)	1822.243 (13)	0.34 (4)	5.0
f <sub>2</sub>	553.725 (22)	1805.95 (7)	0.67 (7)	11.2	553.701 (26)	1806.026 (08)	0.42 (4)	7.6
f <sub>3</sub>	563.500 (46)	1774.62 (14)	0.37 (7)	5.4				
f <sub>4</sub>	565.800 (32)	1767.41 (10)	0.52 (7)	7.6	565.929 (13)	1767.005 (04)	0.77 (4)	15.1
f <sub>5</sub>	614.268 (25)	1627.96 (7)	0.61 (7)	9.8				
f <sub>6</sub>	619.23 (5)	1614.90 (14)	0.30 (7)	4.6	619.126 (13)	1615.180 (03)	0.81 (4)	15.1
f <sub>7</sub>	624.294 (25)	1601.81 (6)	0.61 (7)	9.9				
f <sub>8</sub>	629.727 (32)	1587.99 (8)	0.48 (7)	7.7	630.058 (29)	1587.154 (07)	0.33 (4)	6.8
f <sub>9</sub>	634.850 (15)	1575.175 (37)	1.00 (7)	16.7				
f <sub>10</sub>	639.872 (10)	1562.813 (25)	1.44 (7)	24.4	639.793 (19)	1563.004 (04)	0.51 (4)	10.2
f <sub>11</sub>	641.829 (18)	1558.048 (45)	0.75 (7)	13.3	642.042 (13)	1557.529 (03)	0.81 (4)	14.4
f <sub>12</sub>	646.78 (5)	1546.13 (12)	0.32 (7)	4.9				
f <sub>13</sub>					656.412 (43)	1523.431 (10)	0.33 (4)	4.6
f <sub>14</sub>	661.711 (25)	1511.23 (6)	0.51 (7)	9.9	661.506 (20)	1511.700 (04)	0.52 (4)	9.8
f <sub>15</sub>	664.997 (15)	1503.766 (34)	0.98 (7)	16.4	665.305 (09)	1503.068 (02)	1.12 (4)	21.7
f <sub>16</sub>					679.693 (08)	1471.251 (01)	1.19 (4)	23.1
f <sub>17</sub>	684.955 (27)	1459.95 (6)	0.52 (7)	8.9	684.795 (15)	1460.289 (03)	0.74 (4)	13.3

$$\chi^2(M_\star, T_{\text{eff}}) = \frac{1}{m} \sum_{i=1}^m \min \left[ \left( \Pi_i^{\text{O}} - \Pi_k^{\text{T}} \right)^2 \right], \quad (5)$$

where  $m$  is the number of observed periods. The PG 1159 model that shows the lowest value of  $\chi^2$ , if exists, is adopted as the “best-fit model”. We assess the function  $\chi^2 = \chi^2(M_\star, T_{\text{eff}})$  for stellar masses of 0.515, 0.530, 0.542, 0.565, 0.589, 0.609, 0.664, 0.741, and 0.870  $M_\odot$ . For the effective temperature, we employ a much finer grid ( $\Delta T_{\text{eff}} = 10 - 30$  K) which is given by the time step adopted by our evolutionary calculations.

The mentioned methods to estimate the stellar mass and the internal structure of NGC 246 are the same ones we employed in our previous works at the La Plata Stellar Evolution and Pulsation Research Group<sup>5</sup>, for instance, Córscico et al. (2007a,b, 2008, 2009); Kepler et al. (2014); Calcaferro et al. (2016); Córscico et al. (2021); Uzungdag et al. (2021), where further details can be found.

### 3.1. Period spacing

As a first step, we aim to estimate a mean period separation underlying the observed periods (if it exists). Considering the set of periods from Table 1, we searched for a constant period spacing within the data of NGC 246 employing the Kolmogorov-Smirnov (KS; see Kawaler 1988), the inverse variance (I-V; see O’Donoghue 1994), and the Fourier transform (F-T; see Handler et al. 1997) significant tests. Given that, in principle, there are two different sets of periods coming from two sectors, we followed this procedure for each set of periods separately. We additionally considered a combination of both sets, taking the average value for those cases where two periods are identified as the same mode. We show the resulting list of 17 periods in

<sup>5</sup> <http://evolgroup.fcaglp.unlp.edu.ar/publications.html>

the first column of Table 2. Since we found similar solutions for the three cases, we only show the results for the case of the 17 periods from the combined sectors (Table 2) in Fig. 5, indicated with dotted black lines. The figure shows a clear signature of a potential mean period spacing around 12.9 s, while other possible values are around 10.6, 14.5, and 51 s. Next, we repeated this analysis but employing other sets of periods. In our trials, we found that, for instance, when discarding the periods at around 1503, 1511, 1557, 1562, 1767, and 1774 s, the three statistical tests applied to the resulting set of 11 periods show more robust evidence of a mean period spacing at  $\sim 12.9$  s, as depicted with solid orange lines by Fig. 5, while the other possible values show much lower (or no) statistical significance, hence, we disregard them. Also, the addition or removal of some of these excluded periods gives approximately the same results, in some cases even improving the statistical significance of the tests. In this way, we can adopt the value of  $\sim 12.9$  s as a guess value for the mean period spacing.

We cannot know in advance if this possible mean period spacing is associated with  $\ell = 1$  or 2 modes. We might assume it is a period spacing of  $\ell = 2$  modes for its short value, but then a period spacing of 22.3 ( $= 12.9 \times \sqrt{3}$ , according to Eq. (3)) corresponding to  $\ell = 1$  modes would be expected, which is absent. However, there is no reason to discard this 12.9 s period spacing for a sequence of  $\ell = 2$  modes. In the following section, we estimate the stellar mass of NGC 246 by considering the possibility that  $\Delta\Pi = 12.9$  s is associated with  $\ell = 1$  or  $\ell = 2$  modes.

To derive a refined value of the period spacing first we carried out a linear least-squares fit, using the reduced set of 11 periods. We obtained  $\Delta\Pi = 12.9344 \pm 0.05381$  s. The average value of the residuals ( $\delta\Pi$ ) resulting from the difference between the observed periods and those calculated from the mean period spacing is 1.093 s. We repeated the linear least-squares fit, but this time, including three of the previously discarded periods (1511, 1562, and 1767 s; that is, we used the 14 periods marked with an asterisk in Table 2) that are compatible with the previous lin-

**Table 2.** List of the 17 periods combined from Sectors 3 and 30 for NGC 246.

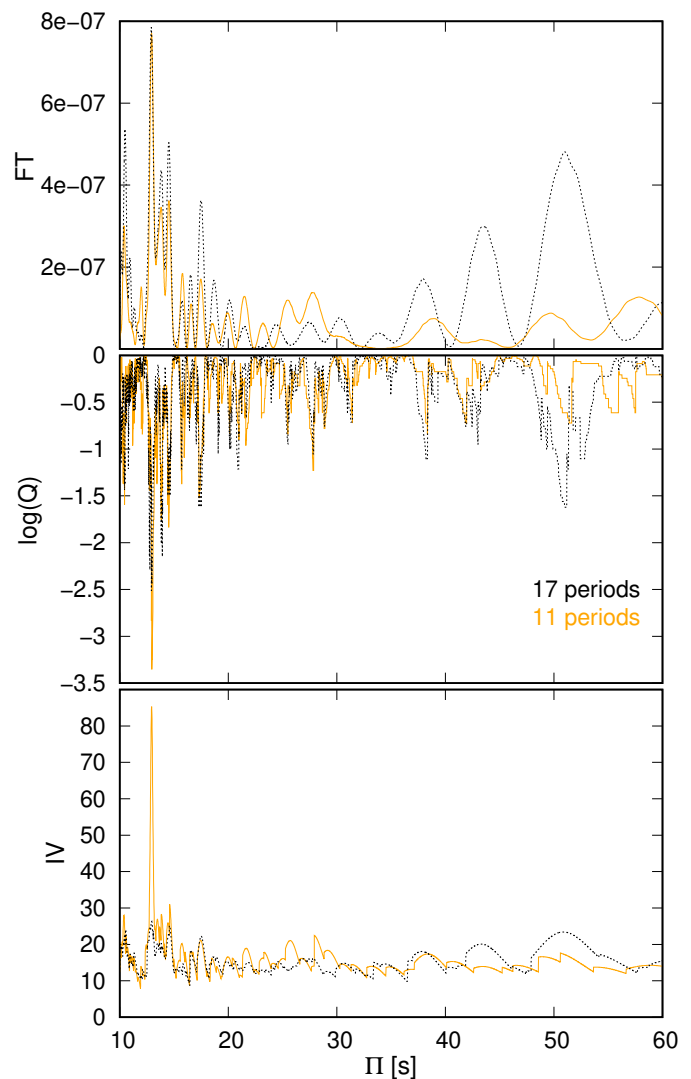
$\Pi^O$ [s]	$\Pi^T$ [s]	$\delta\Pi_{\text{fit}}$ [s]	$\ell$
1460.1195 <sup>(*)</sup>	1459.08	1.0395	1
1471.2515 <sup>(*)</sup>	1471.9818	-0.7308	1
1503.417		?	?
1511.4655 <sup>(*)</sup>	1510.6872	0.7778	1
1523.4315 <sup>(*)</sup>	1523.589	-0.158	1
1546.135 <sup>(*)</sup>	1549.3926	-3.2626	1
1557.7885		?	?
1562.90855 <sup>(*)</sup>	1562.2944	0.6141	1
1575.1755 <sup>(*)</sup>	1575.1962	-0.0212	1
1587.5725 <sup>(*)</sup>	1588.098	-0.526	1
1601.815 <sup>(*)</sup>	1600.9998	0.8102	1
1615.045 <sup>(*)</sup>	1613.9016	1.1384	1
1627.965 <sup>(*)</sup>	1626.8034	1.1566	1
1767.20755 <sup>(*)</sup>	1768.7232	-1.5157	1
1774.62		?	?
1805.9885 <sup>(*)</sup>	1807.4286	-1.4406	1
1822.45655 <sup>(*)</sup>	1820.3304	2.1261	1

**Notes.** The periods marked with an asterisk are used for the linear least-square fit of Fig. 6. The  $\ell$  identification is based on the period spacing extracted from the list of periods (see text for details).

ear fit, and we obtained  $\Delta\Pi = 12.9018 \pm 0.0451$  s. This value of the period spacing is slightly shorter than the one derived previously, and the uncertainty is slightly smaller. Additionally, the average of the residuals is smaller, 0.986 s, so the fitted periods match the observed periods better than before, and this is why we adopt  $\Delta\Pi = 12.9018 \pm 0.0451$  s as the definitive value for the mean period spacing of NGC 246. In Table 2 we indicate the theoretical periods from the fitted mean period spacing (second column) and the corresponding residuals (third column). In the last column, we indicate the possible identification of the fitted periods with  $\ell = 1$  modes, according to the period spacing previously inferred, although they may all be associated with  $\ell = 2$ , as already mentioned. In addition, we show the corresponding linear fit in the upper panel of Fig. 6, while the residuals for this case are displayed in the lower panel. It is worth noticing the presence of some minima in the distribution of the residuals, which can be attributed to the effects of mode trapping caused by the existence of internal chemical transition regions (Kawaler & Bradley 1994).

### 3.2. Mass determination from the comparison between the observed mean period spacing and the average of the computed period spacings

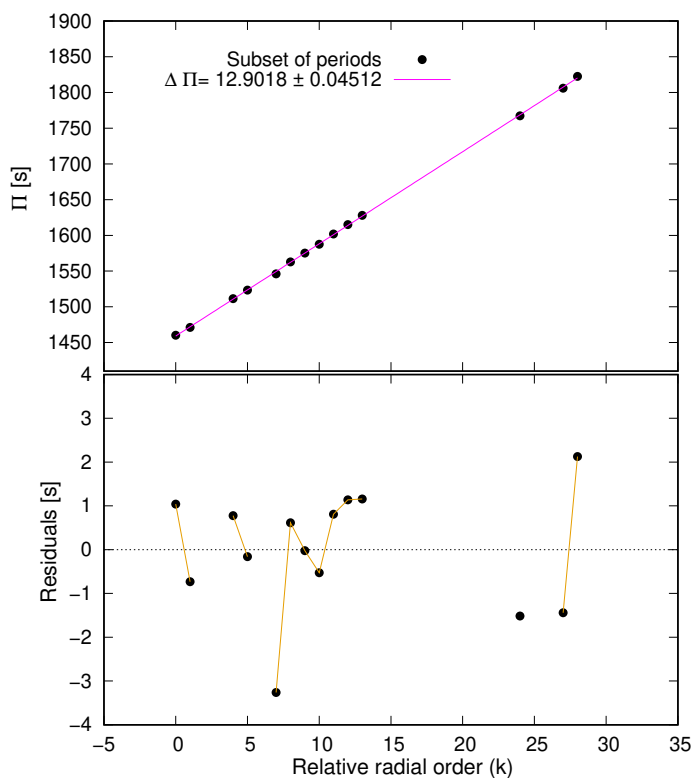
Now, we determine the mass of NGC 246 by comparing the average of the computed period spacings,  $\overline{\Delta\Pi}_k$ , for our grid of models, with the observed period spacing,  $\Delta\Pi$ , as determined in Sect. 3.1. The average of the computed period spacings is calculated as  $\overline{\Delta\Pi}_k = (n - 1)^{-1} \sum_k \Delta\Pi_k$ , where the “forward” period spacing is defined as  $\Delta\Pi_k = \Pi_{k+1} - \Pi_k$  ( $k$  being the radial order) and  $n$  is the number of theoretical periods within the range of the periods observed in the target star. For this star,  $\Pi_k \in [1459, 1823]$  s.



**Fig. 5.** F-T (upper panel), K-S (middle panel) and I-V (bottom panel) significance tests applied to the period spectrum of NGC 246 to search for a constant period spacing. Dotted black lines represent the results for the set of 17 periods from Table 2, while solid orange lines, the case for a subset of 11 periods (see text for details).

In Fig. 7 we show the run of the average of the computed period spacings for NGC 246 for the case in which the period spacing is assumed to be associated with  $\ell = 1$  modes, indicated with thin curves, and  $\ell = 2$  modes, represented with thick curves, in terms of  $T_{\text{eff}}$  for our PG 1159 evolutionary sequences. The location of NGC 246 in this  $\overline{\Delta\Pi} - T_{\text{eff}}$  plane is represented by a black circle with error bars, where we used the period spacing (and its uncertainty) determined in Sect. 3.1, and the star effective temperature (with a 10% for its uncertainty, that is, slightly larger than the one given by the latest spectroscopic results). Clearly, for each set of curves with  $\ell = 1$  and 2, the lower the values of  $\overline{\Delta\Pi}_k$ , the greater the stellar mass. When we consider the measured period spacing  $\Delta\Pi = 12.9018 \pm 0.0451$  s to be associated with  $\ell = 1$  modes, it is clear that the stellar mass is slightly larger than  $0.87 M_{\odot}$ . If we, instead, consider that the period spacing corresponds to  $\ell = 2$  modes, a linear interpolation between  $\Delta\Pi$  and  $\overline{\Delta\Pi}_k$ <sup>6</sup> yields a stellar mass of  $M_{\star} = 0.568^{+0.006}_{-0.012} M_{\odot}$ .

<sup>6</sup> When there were no points to perform the linear interpolation, we extrapolated the theoretical values of  $\overline{\Delta\Pi}_k$ .

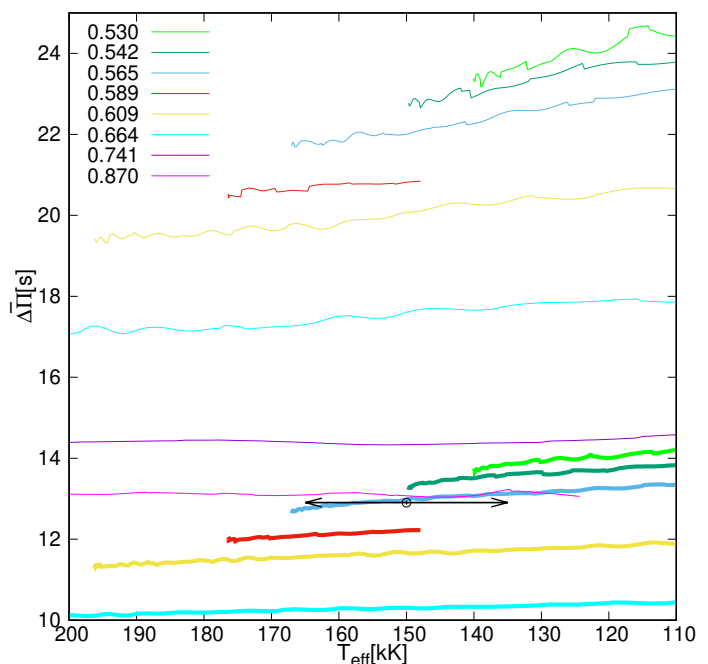


**Fig. 6.** *Upper panel:* Linear least-squares fit applied to a subset of 14 periods of NGC 246 (see text for details). The resulting period spacing is  $\Delta\Pi = 12.9018 \pm 0.04512$  s. *Lower panel:* Residuals of the period distribution relative to the dipole mean period spacing. A thin orange line connects modes with consecutive radial order.

### 3.3. Constraints from the individual observed periods: searching for the best-fit model

To carry this procedure out, we took different sets of periods into account: each set of periods from Sectors 3 and 30 from Table 1, and the combination of both, as indicated in Table 2. In this last case, we additionally considered two particular subsets of the combination: one subset with 11 periods (see Sect. 2.2), and another one with the 14 periods marked with asterisks in Table 2, that fit the mean period spacing determined in Sect. 3.1. For all cases, we first considered that all the periods are associated with  $\ell = 1$   $g$  modes and employed them to assess the quality function given by Eq. (5). Next, we repeated the procedure, but considering they are associated with a mixture of  $\ell = 1$  and  $\ell = 2$   $g$  modes. Generally, if a single maximum exists, we can adopt the corresponding model as the asteroseismic solution. Unfortunately, in the cases we studied here there are multiple possible solutions, and then, we need to apply an external constraint in order to adopt a single asteroseismic model. In this case, the constraint is the effective temperature and its uncertainty (where, again, we took a more flexible value than the one given by spectroscopy).

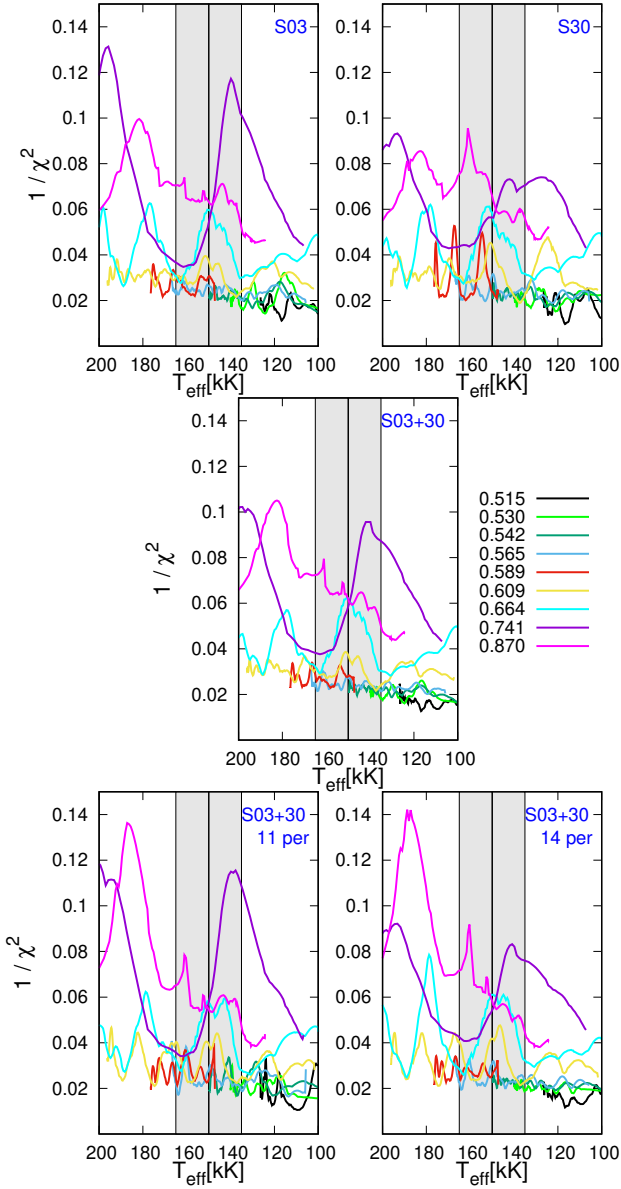
We start by considering the list of 15 periods from Sector 3 (Table 1), and we show the results for the  $\ell = 1$  case in the upper left panel of Fig. 8. In the figure, we plot the inverse of the quality function,  $1/\chi^2$ , such that the greater the value of  $1/\chi^2$ , the better the fit between theoretical and observed periods. It is clear that there is more than one possible asteroseismic solution. In particular, the absolute maximum (the best solution) in this case corresponds to a model with  $T_{\text{eff}} \sim 196\,000$  K, which is much higher than the values of  $T_{\text{eff}}$  allowed by the spectroscopic deter-



**Fig. 7.** Average of the computed period spacings,  $\overline{\Delta\Pi}_k$ , assessed in the range of periods observed in NGC 246, corresponding to our PG 1159 evolutionary sequences (with stellar masses expressed in solar units), in terms of the effective temperature. Thin curves represent the case in which the period spacing is assumed to be associated with  $\ell = 1$  modes, while thick curves, with  $\ell = 2$  modes. The location of NGC 246 ( $\Delta\Pi = 12.9018 \pm 0.0451$  s,  $T_{\text{eff}} = 150\,000 \pm 15\,000$  K) is emphasized using a black circle with error bars.

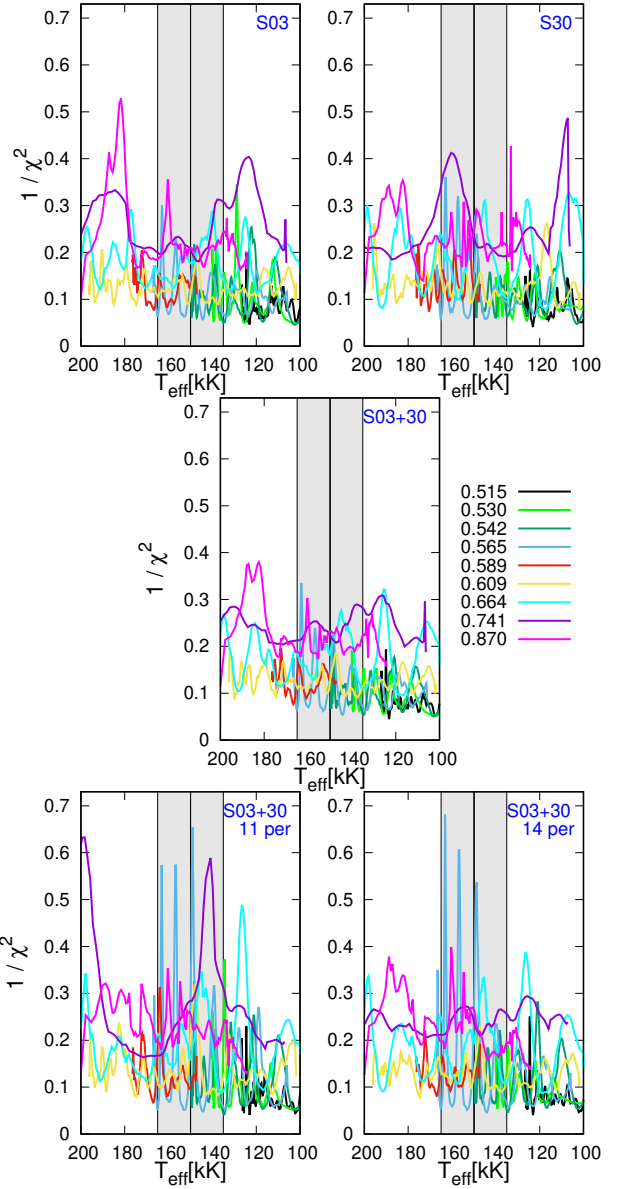
minations for NGC 246, and can be discarded. The second-best global fit lies within the constraints given by the allowed range of effective temperatures at  $\sim 140\,000$  K, for  $M_{\star} = 0.741 M_{\odot}$  and  $1/\chi^2 = 0.117$ . When we consider the list of 12 periods from Sector 30 (Table 1), and repeat the above procedure, we obtain the results shown in the upper right panel of Fig. 8. Once again, there are multiple possible solutions, but the best-global fit lies within the range of allowed  $T_{\text{eff}}$  at  $\sim 160\,000$  K, for  $0.87 M_{\odot}$  and  $1/\chi^2 = 0.096$ . Next, we consider the set of 17 periods combining Sectors 3 and 30, as listed in Table 2. The results are shown in the middle panel of Fig. 8. The best-global fit lies at a very high  $T_{\text{eff}}$ , and within the ranges of allowed  $T_{\text{eff}}$ , there is a possible solution characterized by  $\sim 140\,000$  K, for  $0.741 M_{\odot}$  and  $1/\chi^2 = 0.097$ . The lower left panel of Fig. 8 depicts the results when considering the subset of 11 periods. In this case, the best-global fit lies at high values of  $T_{\text{eff}}$ . However, in the allowed ranges of  $T_{\text{eff}}$ , there is a possible solution characterized by  $\sim 140\,000$  K, for  $0.741 M_{\odot}$  and  $1/\chi^2 = 0.116$ . The results for the subset 14 periods are shown in the lower right panel of Fig. 8. Similarly to the previous case, although the best-global fits lie at very high values of  $T_{\text{eff}}$ , there are some possible solutions, with comparable quality, given by  $\sim 160\,000$  K, for  $0.870 M_{\odot}$  and  $1/\chi^2 = 0.092$  and  $\sim 140\,000$  K, for  $0.741 M_{\odot}$  and  $1/\chi^2 = 0.083$ .

Regarding the  $\ell = 1, 2$  case, we show the results in Fig. 9. Clearly, the quality of the solutions is in general improved, as evidenced by the greater values of  $1/\chi^2$ . The results for the set of periods from Sector 3 are shown in the upper left panel. The best global fits are located at higher and lower values of  $T_{\text{eff}}$  than those given by spectroscopy. However, there are possible solutions within the allowed ranges of  $T_{\text{eff}}$  at  $\sim 160\,000$  K, for



**Fig. 8.** Inverse of the quality function,  $1/\chi^2$ , versus  $T_{\text{eff}}$  of the period-to-period fits in the  $\ell = 1$  case, for the five sets of periods considered for NGC 246: Sector 3 (*upper left panel*), Sector 30 (*upper right panel*), the 17 periods from the combination of both sectors (*middle panel*), and the two subsets of the combination of both sectors with 11 (*lower left panel*) and 14 periods (*lower right panel*). The vertical gray strip indicates the spectroscopic values of  $T_{\text{eff}}$  and the corresponding uncertainty for this star.

0.870  $M_{\odot}$  and  $1/\chi^2 = 0.356$ , at  $\sim 140\,000$  K, for 0.741  $M_{\odot}$  and  $1/\chi^2 = 0.313$ , and also at  $\sim 163\,000$  K, for 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.301$ . For Sector 30, as shown in the upper right panel of Fig. 9, the best solution has a very low  $T_{\text{eff}}$ , and the solutions that lie within the range of allowed  $T_{\text{eff}}$  are characterized by  $\sim 160\,000$  K, 0.741  $M_{\odot}$ , and  $1/\chi^2 = 0.412$ , and  $\sim 163\,000$  K, 0.565  $M_{\odot}$ , and  $1/\chi^2 = 0.361$ . For the combined set of periods from both sectors, as the middle panel of Fig. 9 depicts, the best-fit models have very high values of  $T_{\text{eff}}$ . Still, there are three possible solutions within the range of allowed  $T_{\text{eff}}$  that are worth mentioning at  $\sim 163\,000$  K, for 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.335$ , at  $\sim 160\,000$  K, for 0.870  $M_{\odot}$  and  $1/\chi^2 = 0.303$ , and also at  $\sim 140\,000$  K, for 0.741  $M_{\odot}$  and  $1/\chi^2 = 0.289$ . The case of the subset with 11 periods, represented in the lower left panel



**Fig. 9.** Same as 8, but for the  $\ell = 1, 2$  case.

of Fig. 9, shows the best-global fit for a model characterized by  $\sim 150\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.654$ . Other possible solutions within the range of allowed  $T_{\text{eff}}$  are characterized by  $\sim 140\,000$  K, 0.741  $M_{\odot}$  and  $1/\chi^2 = 0.589$ ,  $\sim 157\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.573$ , and  $\sim 163\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.574$ . Finally, the lower right panel of Fig. 9 depicts the results for the subset with 14 periods. In this case, the best-global fit is represented by a model with  $\sim 163\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.681$ . It is worth mentioning that this solution has the best quality (the highest value of  $1/\chi^2$ ) among all the cases considered in this work. Other possible solutions, that also lie within the range of allowed  $T_{\text{eff}}$ , are characterized by models with  $\sim 157\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.607$ , and  $\sim 150\,000$  K, 0.565  $M_{\odot}$  and  $1/\chi^2 = 0.537$ .

When we compare the results from all cases, it is clear that the global behavior of the quality function changes, although some solutions are repeated in more than one case. Within the allowed ranges of  $T_{\text{eff}}$  four solutions stand out, in general, corresponding to the asteroseismic models with 0.741  $M_{\odot}$  and  $\sim 140\,000$  K, 0.870  $M_{\odot}$  and  $\sim 160\,000$  K, when we only use

$\ell = 1$  modes for the fits and when we use a mixture of  $\ell = 1, 2$  modes, and also the models with  $0.565 M_{\odot}$  and  $\sim 163\,000$  K, and  $0.565 M_{\odot}$  and  $\sim 150\,000$  K in the case of  $\ell = 1, 2$ .

Given these four best-fitting models, we calculated their asteroseismic distances and compared them with the distance resulting from the *Gaia* measurements. This is analogous to the procedure followed in the previous works of this series (see also Bell et al. 2019; Uzundag et al. 2023). Considering the luminosity of the four asteroseismic models, namely:  $(\log(L_{\star}/L_{\odot}) \sim 4.25, M_{\star}[M_{\odot}] = 0.741, T_{\text{eff}}[K] \sim 140\,000)$ ,  $(4.51, 0.870, \sim 160\,000)$ ,  $(3.37, 0.565, \sim 163\,000)$ , and  $(3.55, 0.565, \sim 150\,000)$ , and using the bolometric correction for the  $T_{\text{eff}}$  of the different models (interpolated from the works of Kawaler & Bradley 1994; Córscico et al. 2007a; Uzundag et al. 2021), then the visual absolute magnitude is calculated as  $M_V = M_B - BC$ , where  $M_B = M_{B\odot} - 2.5 \times \log(L_{\star}/L_{\odot})$  and  $M_{B\odot} = 4.74$  (Cox 2000). In this way, the asteroseismic distance,  $d$ , can be inferred from  $\log(d) = [m_V - M_V + 5 - A_V(d)]/5$ . The apparent visual magnitude,  $m_V$ , according to the *TESS* catalog, is 12.3061 mag, while the interstellar absorption  $A_V(d) = R_V E(B - V)$  (with  $R_V = 3.1$ ) for this star is 0.062 mag, according to Frew et al. (2016) and Ali et al. (2023). From this, the resulting distances are, approximately, 1279, 1473, 395, and 525 pc for the models with  $0.741, 0.870$ , and  $0.565 M_{\odot}$  with  $\sim 163\,000$  K and  $\sim 150\,000$  K, respectively. We have also computed the distances for other reported values of the interstellar absorption for this star, that is,  $A_V = 0.14$  mag (González-Santamaría et al. 2021), and  $A_V = 0.0806$  mag, that results from the value of  $E(B - V)$  obtained from the 3D reddening map (Lallement et al. 2014; Capitanio et al. 2017; Lallement et al. 2018)<sup>7</sup>, according to the galactic coordinates of this star,  $(l, b) = (118.^{\circ}8630881437, -74.^{\circ}7090941475)$ . However, the use of these alternative values for  $A_V$  does not significantly change the resulting distance estimations (the difference being within some tens of pc). Considering the distance estimate from *Gaia* EDR3,  $d_G = 556^{+25}_{-23}$  pc (Gaia Collaboration 2020), it is clear that the model with  $0.565 M_{\odot}$  and  $\sim 150\,000$  K has its distance in line with this value. In this way, although this possible asteroseismic solution is not unique, it represents a good period-to-period fit, with a distance compatible with the precise distance estimate from *Gaia* for NGC 246. It is also worthwhile to mention that an independent distance measurement was performed by Bond & Ciardullo (1999). Fitting the dwarf companion to the zero-age main sequence, they derived a distance of  $495^{+145}_{-100}$  pc, which is in agreement with the *Gaia* distance.

Finally, a calculation of the stellar mass when fixing the distance of NGC 246 to the value given by *Gaia* for different values of  $A_V$  and some effective temperatures in the range of interest, results in model sequences characterized by a stellar mass around  $0.6 M_{\odot}$ . For the resulting luminosities to be compatible with stellar masses  $\geq 0.7 M_{\odot}$ ,  $A_V$  should be  $\geq 1.5$  mag, which, then, would be too large and inconsistent with the high value of the galactic latitude of NGC 246 below the Galactic plane ( $b = -74.^{\circ}7090941475$ ).

#### 4. Summary and discussion

In this paper, we have presented a detailed asteroseismic study of NGC 246, a pulsating PG 1159 star, based on the high-precision photometry data from *TESS* observations. NGC 246 ( $T_{\text{eff}} \sim 150\,000$  K and  $\log g \sim 5.7$ ) was observed by *TESS* in Sector 3, at the short 120 s cadence, and 30, at the short 120 s and ultra-short 20 s cadence. Our frequency analysis revealed a total

of 17 periodicities. These oscillation periods (1460 - 1823 s) are associated with nonradial  $g$  modes. Given this frequency spectrum, we investigated the presence of rotation multiplets. However, we did not find any distinct indications of rotational splitting for either dipole or quadrupole modes.

Employing the inferred pulsation periods, we carried out an asteroseismic analysis on NGC 246 by means of our fully evolutionary models of PG 1159 stars. We first found a constant period spacing underlying the observed periods, which was used to make inferences on the stellar mass via the comparison between the observed mean period spacing and the average of the computed period spacings. Our results indicate that if the observed mean period spacing is associated with  $\ell = 1$   $g$  modes, then  $M_{\star} \geq 0.87 M_{\odot}$ . If, instead, it is associated with  $\ell = 2$   $g$  modes, then  $M_{\star} \geq 0.568 M_{\odot}$ . Next, we searched for the best-fit model between the theoretical and the observed periods. For this procedure, we first assumed that all the periods were associated with  $\ell = 1$   $g$  modes only, and subsequently, with a mixture of  $\ell = 1, 2$   $g$  modes. Although we did not find a clear and unambiguous solution for NGC 246, there are some worth mentioning possible solutions that lie within the values of effective temperature (for which we took an error of 10%). When considering that all modes are  $\ell = 1$  only, our results suggest a high-stellar mass ( $\geq 0.74 M_{\odot}$ ), but when we allow the modes to take a mixture of  $\ell = 1$  and 2, both high ( $\geq 0.74 M_{\odot}$ ) and intermediate ( $\sim 0.57 M_{\odot}$ ) stellar masses are possible. A comparison between the corresponding asteroseismic distances for these models to the *Gaia* astrometric distance ( $556^{+25}_{-23}$  pc, Gaia Collaboration 2020) indicates that only one solution is in agreement, which corresponds to a model with  $M_{\star} = 0.565 M_{\odot}$ , lying at approximately the same  $T_{\text{eff}}$  of the star.

NGC 246 has been recently analyzed by Löbbling (2018, 2020) who, employing new spectroscopic observations and state-of-the-art non-LTE models, found  $T_{\text{eff}} = 150\,000 \pm 10\,000$  K and  $\log g = 5.7 \pm 0.1$ . This results in  $M_{\star} = 0.74^{+0.19}_{-0.23} M_{\odot}$  and then, in a stellar radius of  $R_{\star} = 0.20 R_{\odot}$ . Interestingly, in the spectral analysis carried out by Rauch & Werner (1997), the authors noted unusually broad spectral line cores, which was interpreted as a signature of stellar rotation with velocity  $v \sin i \approx 70$  km s<sup>-1</sup>, being  $i$  the unknown inclination angle of the rotation axis. This value was later confirmed by using high-resolution spectra by Rauch et al. (2004). More recently, Löbbling (2018) measured  $v \sin i = 75 \pm 15$  km s<sup>-1</sup>. Considering the latter along with the stellar radius derived from spectroscopy, it would imply a rotational period of  $P \times \sin i = 0.68 \pm 0.14$  d. If this were the rotational period that characterizes NGC 246, a rotational splitting of  $8.51 \mu\text{Hz}$  would be expected in the frequency spectrum in  $\ell = 1$  modes. However, neither the amplitude spectra of Sectors 3 and 30 exhibit a clear pattern that would confirm this expectation. Still, it must be noted that the rotational velocity could be overestimated because, e.g., microturbulent motions in the atmosphere could contribute to the line-core broadening. Additionally, Löbbling (2020) determined a spectroscopic distance of  $970 \pm 200$  pc for NGC 246. This is a factor of 1.7 larger than the *Gaia* distance. In order to bring the spectroscopic distance in agreement with the parallax distance, an increase of the surface gravity by a factor of three (that is, an increase of 0.5 dex in  $\log g$ ) would be necessary. As demonstrated by Löbbling (2020), such a high value for the gravity strongly contradicts the model fits to the He II line profiles. The reason for this problem remains unknown. Interestingly, our possible asteroseismic solution with  $0.565 M_{\odot}$  and  $\sim 150\,000$  K, has a value of  $\log g (= 6.29)$ , which is in line with the value needed to fit the parallax distance.

<sup>7</sup> <https://stilism.obspm.fr/>

All in all, our results show two possible outcomes in the derivation of the stellar mass. The use of the pulsation periods via the comparison between the observed period spacing and the average of the computed period spacings points to a very high-stellar mass asteroseismic model if the derived mean period spacing is associated with  $\ell = 1$   $g$  modes, and an intermediate stellar mass if, instead, it is associated with  $\ell = 2$   $g$  modes. Although we were not able to adopt an asteroseismic model from the period-to-period fits, the results obtained from the different sets of periods considered in this procedure would indicate high-stellar mass models for the  $\ell = 1$  case, and both high- and intermediate-stellar mass models for the  $\ell = 1, 2$  case. Astrometry, by means of the precise measurements from Gaia, seems to point toward intermediate stellar masses, making one of our possible asteroseismic solutions particularly promising. At the same time, abiding by the precise distance from Gaia, our asteroseismological results may indicate that the set of modes of this star should be a mixture of  $\ell = 1, 2$ . Regarding the stellar rotation, the conclusions derived from the spectroscopic analysis are not consistent with the precise astrometric distance, an aspect that cannot be ignored. Concurrently, asteroseismology does not so far provide a conclusive value for the rotation period.

Our lack of conclusive results shows the enigmatic nature of NGC 246. Hopefully, future follow-up observations from space, such as the PLAnetary Transits and Oscillations mission (PLATO, [Piotto 2018](#)), or ground-based program, such as BlackGEM ([Bloemen et al. 2015](#)), might help in shedding some light on this open problem. Additionally, improved atmospheric parameters from high-resolution spectroscopic data are necessary to put better constraints on the modeling.

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